Physical Knot Theory

Based on

"A BRIEF INTRO TO KNOT THEORY
FROM THE PHYSICAL POINT OF VIEW"
By Adams

and

"Physical Knot Theory: An Introduction..."
by Millett

Q/ How do we model knots as physical objects?

- Thickness, Length
- Rigid segments / Lattice knots

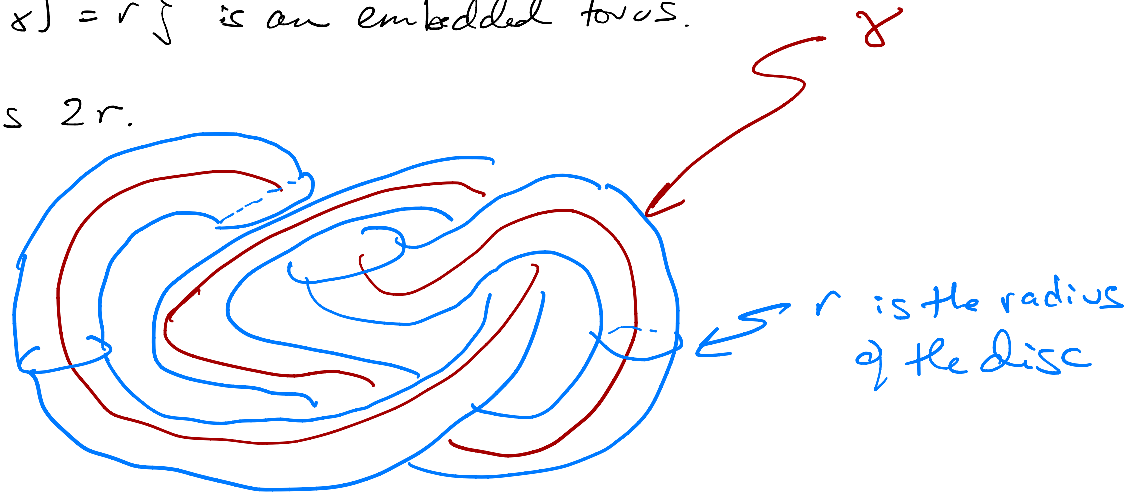
Def Suppose γ is a simple closed curve in $\mathbb{R}^3$

A tubular neighborhood is $\{x \in \mathbb{R}^3 \mid d(x,\gamma) \le r\}$

A *thick knot* is such a tubular neighborhood s.t.

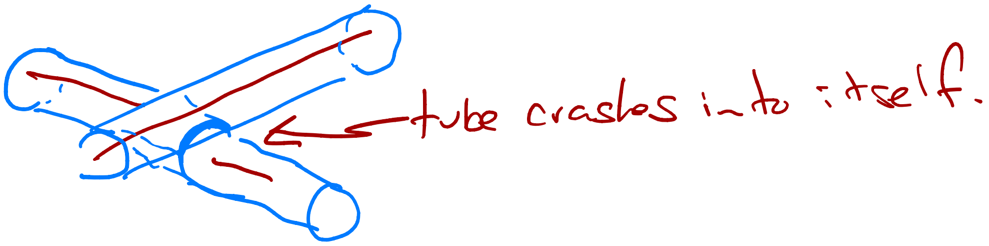
$\{x \in \mathbb{R}^3 \mid d(x,\gamma) = r\}$ is an embedded torus.

Its *thickness* is $2r$.



It's *embedded* if it doesn't intersect itself

eg:



For a knot type K (such as "trefoil")

the *rope length* (or just *length*) of

K is

$$L(K) = \underset{\text{"smallest"}}{\inf}\left\{ \ell \;\middle|\; \begin{array}{l} \exists \text{ a curve } \gamma \text{ in } \mathbb{R}^3 \\ \text{of length } \ell \text{ having} \\ N_{1/2}(\gamma) \text{ embedded} \end{array} \right\}$$

Fact (Buck, Simon, Rawdon)

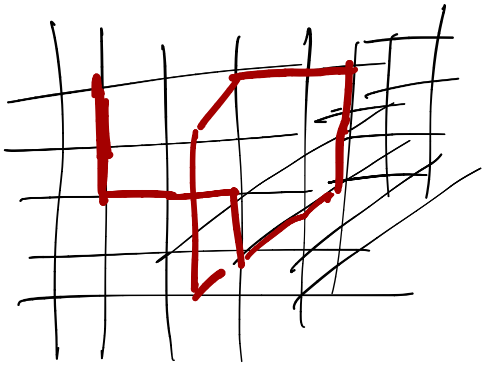
$$(\text{const})\ cr(K)^{3/4} \geq \mathcal{J}(K) \geq 2.135\ cr(K)$$

For <u>NO</u> knot type is the rope length actually known

$\mathcal{J}(\text{trefoil}) \approx 32.7/2$

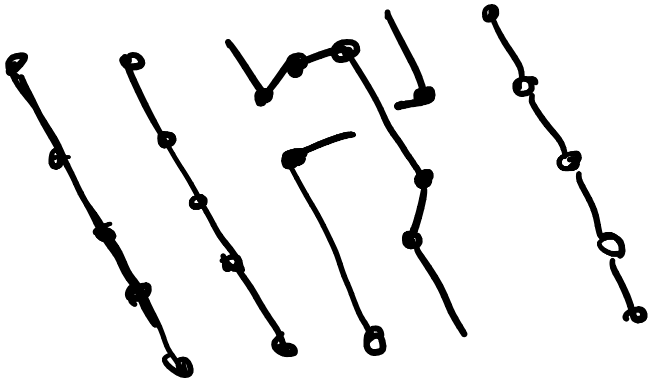
$\mathcal{J}(\text{nontriv knot}) \geq 31.3/2$

One strategy is to deform K to lie on cubic lattice



EASY to estimate length.

Lemma (Diao & Ernst)
If B is a braid w/ b strings & n crossings, then its closure can be realized on cubic lattice w/ at most $12bn$ edges.



$\Rightarrow$ $4(b+1)n$ is length of open braid

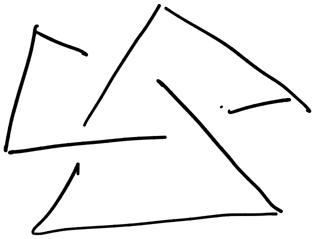
need additional $4n+2$ edges to close for each strand

Can assume $n \geq b-1$

$\Rightarrow$ length $\leq 8bn + 4n + 2b \leq 12bn$

Gives $O(bn)$ as upper bound for $\mathcal{L}$.

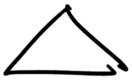
STICK



Model physical knots as polygons in $\mathbb{R}^3$ eg. atoms & bonds.

stick # (knot type) = min # of edges in a polygon having that knot type

stick (unknot) = 3

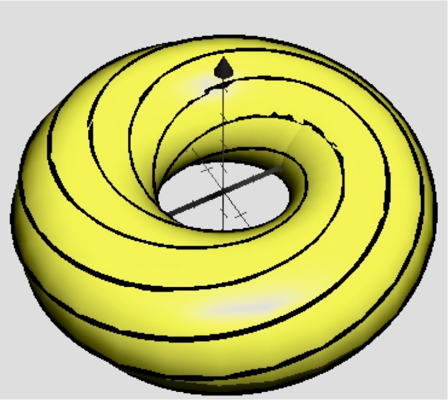


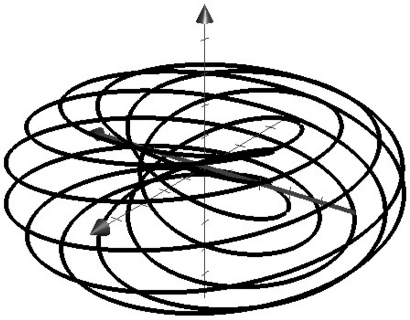
stick (trefoil) = 6

A (p,q) torus knot is a knot lying on a torus wrapping p times one way & q times the other.

It's known that if $p < q < 2p$ then

stick # $= 2q$

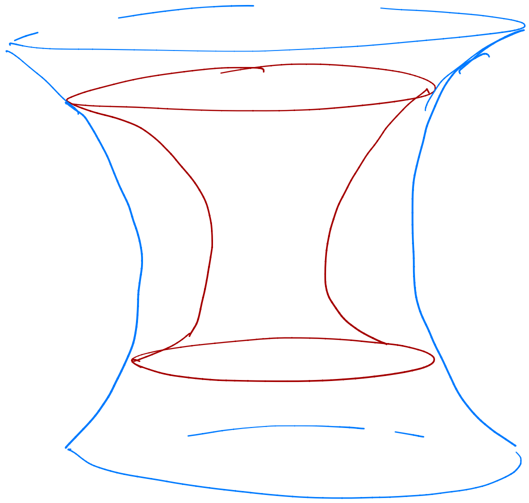




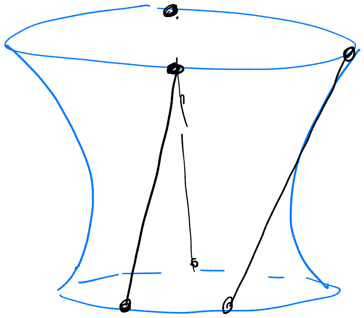
How?

Upper bounds we need a specific configuration

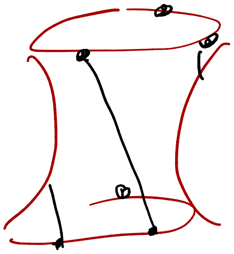
For a torus knot we can build one with two intersecting hyperboloids of different waist sizes

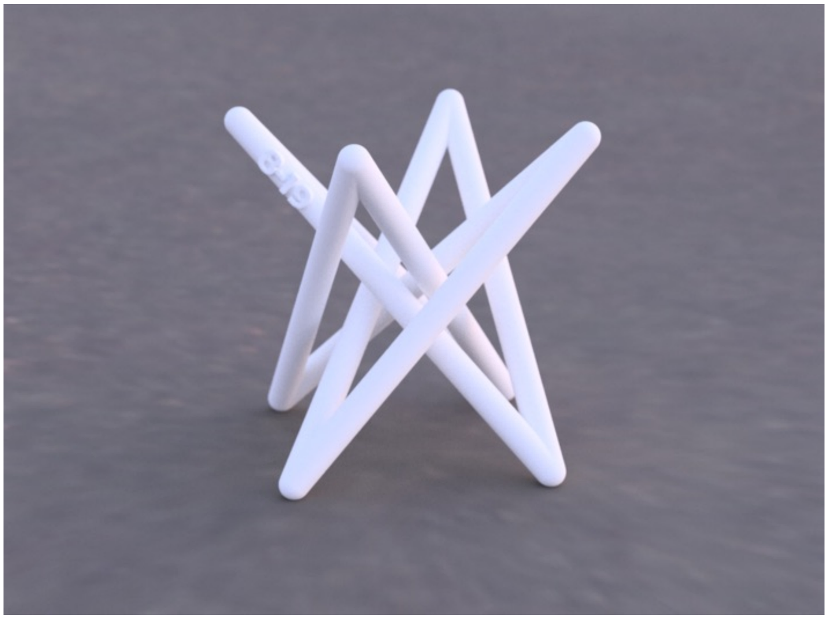


These are ruled surfaces, so we can join points by straight lines lying in the surface (choosing the points carefully)



doing this in both hyperbolas produces a stick configuration of a torus knot.



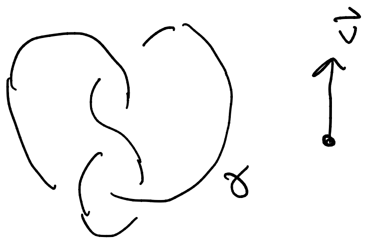


from Shapeways by mathgrrl

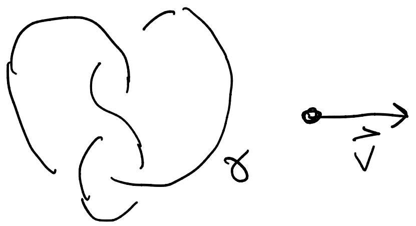
Def If γ is a simple closed curve in $\mathbb{R}^3$ and if $\vec{v} \in \mathbb{R}^3$ is a unit vector (i.e. $\vec{v} \in S^2$) then

$b(\gamma, \vec{v})$ = # of maxima of γ in the direction $\vec{v}$

ex



$b(\gamma, \vec{v}) = 2$



$b(\gamma, \vec{v}) = 3$

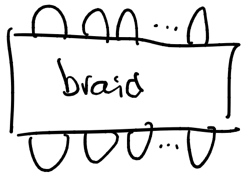
If K is a knot type then $b(K) = \min_{\gamma} \min_{\vec{v} \in S^2} b(\gamma, \vec{v})$

where γ has knot type K.

Ex If $b(K) = 1$ then K is the unknot $\Rightarrow b\begin{pmatrix}\text{Fig-8} \\ \text{Knot}\end{pmatrix} = 2$

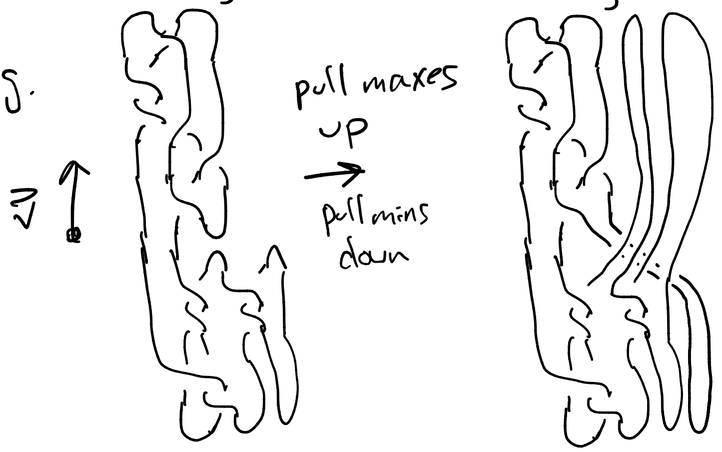
pf. idea $b(K) = 1 \Rightarrow K =$  $= O$ □

Ex If $b(K) = n$ the K has a diagram of the form



pf let $\gamma, \vec{v}$ achieve the min,

eg.



□

Def let γ be a simple closed curve in $\mathbb{R}^3$

its super bridge # is $\quad sb(\gamma) = \max\limits_{\vec{v} \in S^2} \left(\begin{array}{c} \#\text{ maxima in} \\ \text{direction of } \vec{v} \end{array} \right)$

(assuming it is finite
i.e. not

$\uparrow \vec{v}$)

for a knot type K

$$sb(K) = \min_{\gamma} sb(\gamma)$$

s.t. γ has knot type K.

Facts

- $b(K) \underset{\substack{\uparrow \\ \text{obvious}}}{\leq} sb(K) \underset{\substack{\uparrow \\ \text{not obvious}}}{\leq} 2\,b(K)$

- $sb(K) \leq \text{stick}\,\#\,(K)$

How to sample knot space?

Eg. How many distinct knot types represented by <u>equilateral</u> polygons of length n?

★ HOMFLYPT used to distinguish

★ bounded but unknown

★ experimentally many knot types appear only once.

<u>Conjecture</u> (Millett)

On Average, knotting & slipknotting is strongly local

if Average length of knot $\leq$ constant.