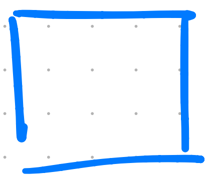
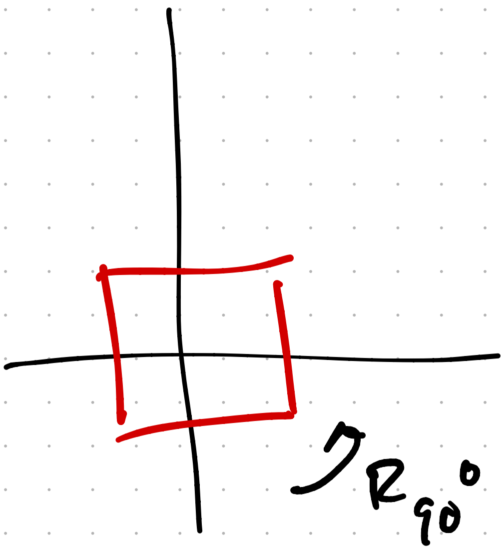
Homomorphisms of Groups

Motivating Ex

$$D_4 = \text{Isom}(\square)$$



$\circlearrowleft R_{90^\circ} \in D_4$



$\circlearrowleft R_{90^\circ}$

$\cap$ (rotated $\in$)

D_4

match up
the elements
preserving combinations

Def If G, G' are groups and if $h: G \to G'$ is a function, then h is a **homomorphism** if $\forall a, b \in G$

$$h(ab) = h(a)h(b)$$

in G in G'

Ex $h: \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$

$$h(a) = a \bmod 5$$

$$\begin{aligned} h(10+29) &= h(39) \\ &= 4 \pmod 5 \\ &= h(10) + h(29) \\ &= 0 + 4 \pmod 5 \end{aligned}$$

Ex $G = \{0, 2, 4, 6, \dots, -2, -4, \dots\}$

$G' = \mathbb{Z}$

$f : G' \to \mathbb{Z}$

$f(a) = 2a$

$0 \to 0$

$2 \to 4$

$4 \to 8$

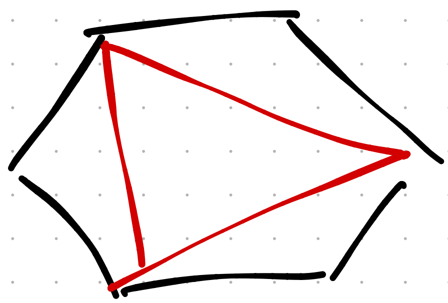
etc.

this is a homorphism.

Ex

$D_3 = \mathrm{Isom}(\triangle)$

$D_6 = \mathrm{Isom}($⬡$)$



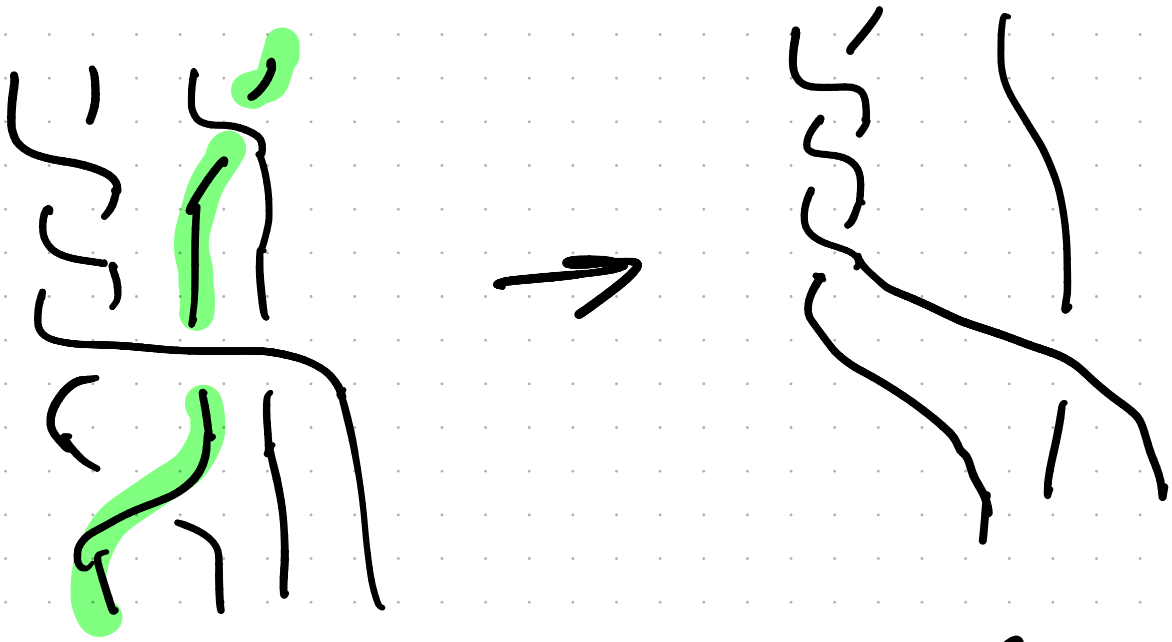
Any symm. of ▷ extends to a symm. of ⬡, so this defines a homomorphism

$$D_3 \to D_6$$

Ex $B_n = \{\text{braids on } n\text{-strands}\}$

define $h: B_n \to B_{n-1}$
by deleting a strand

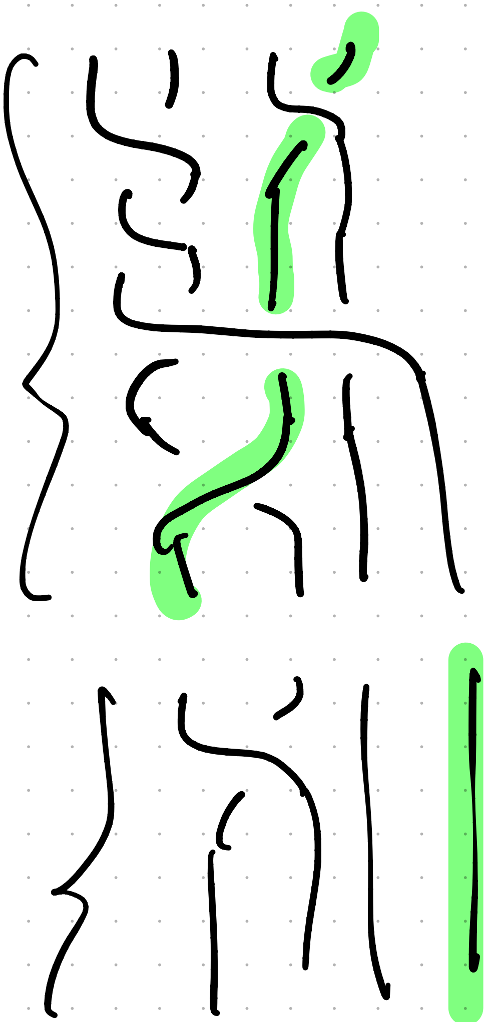
Ex $n = 4$



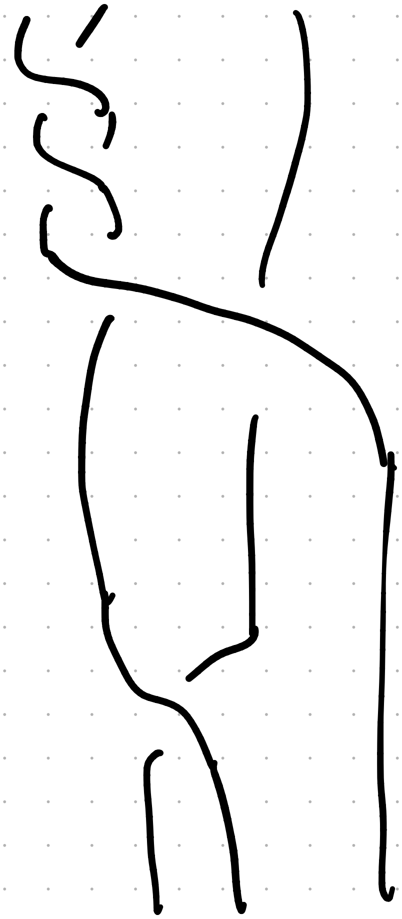
$\uparrow$ B_4 $\qquad\qquad$ $\uparrow$ B_3

A

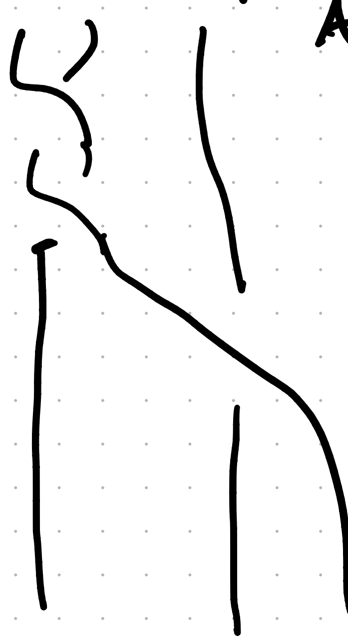
B



$\rightarrow$



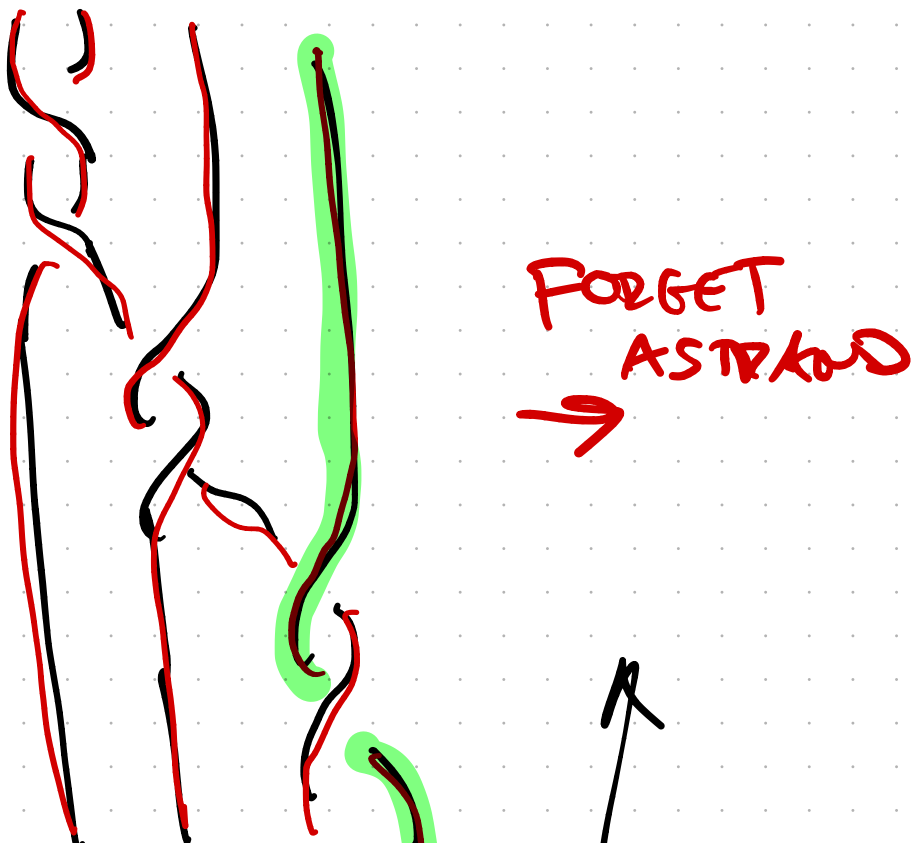
$\neq$



NOT A HOMOMORPHISM

Ex Pure braid group

$PB_n = \{$ n-braids where the ith strand connects the ith starting pt to the ith ending pt $\}$



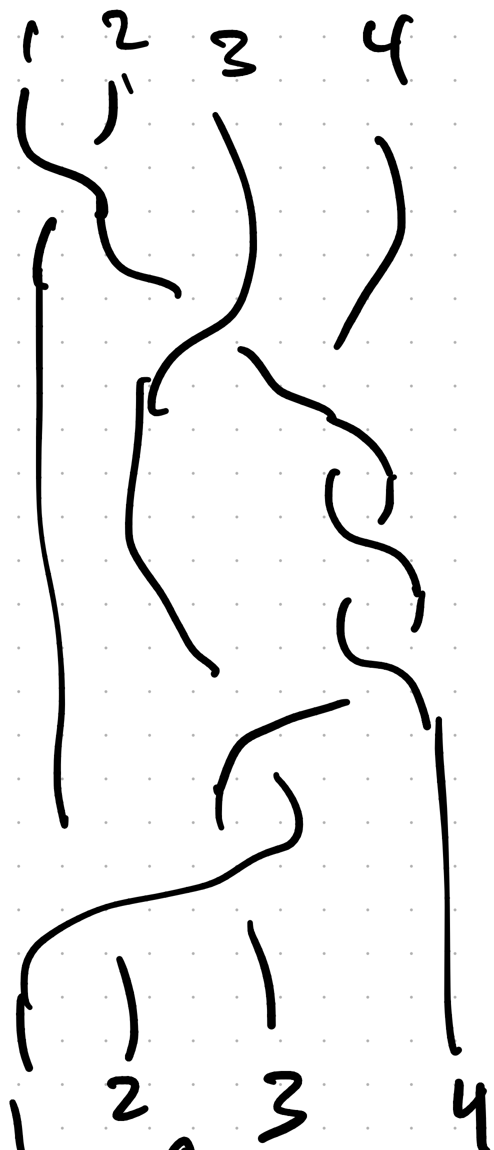
$\cap$
PB_4

homomorphisms

$\cap$
PB_3

Ex Homomorphism from

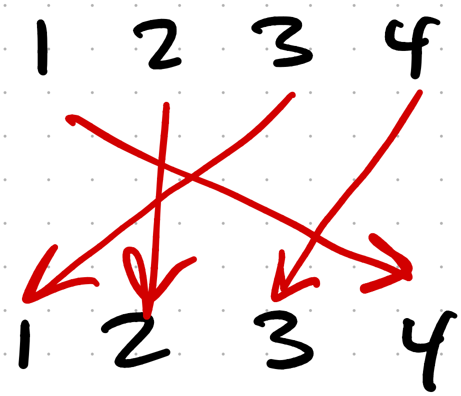
$$B_n \to \text{Permutations of } n \text{ pts} = S_n$$



$\Uparrow$

B_4

$\xrightarrow{h}$



$$PB_n = \{\beta \in B_n \mid h(\beta) = \mathbb{1}\}$$

Ex (Sophisticated)

Recall $F_2 = \{ \text{words in } \{a, b, a^{-1}, b^{-1}\} \}$

w/ $\cdots \cancel{a}\cancel{a^{-1}} \cdots$

$\cdots \cancel{a^{-1}}\cancel{a} \cdots$

$\cdots \cancel{b}\cancel{b^{-1}} \cdots$

$\cdots \cancel{b^{-1}}\cancel{b} \cdots$

operation is concatenation

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}$$

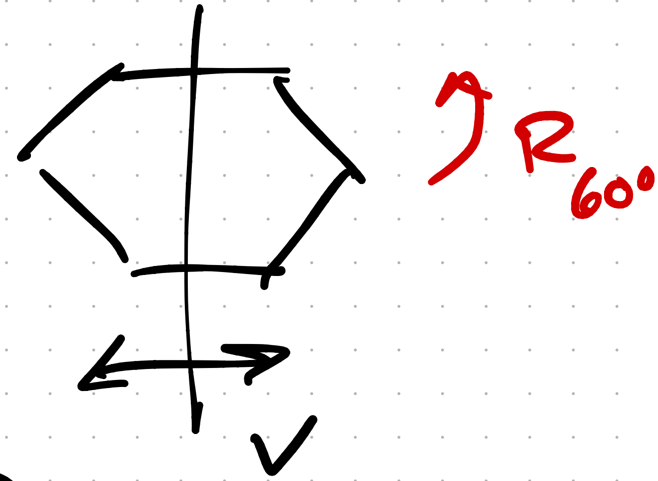
Define $h(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$h(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$h(aba^2) \underset{\text{def}}{=} h(a)\, h(b)\, h(a)^2$$

This is an injective homorphism.

Ex $D_6 =$



Def $h : F_2 \to D_6$

by declaring $h(a) = R_{60^\circ}$

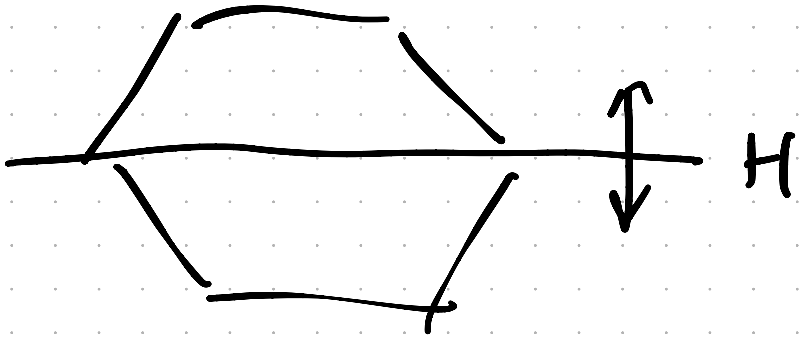
$h(b) = V$

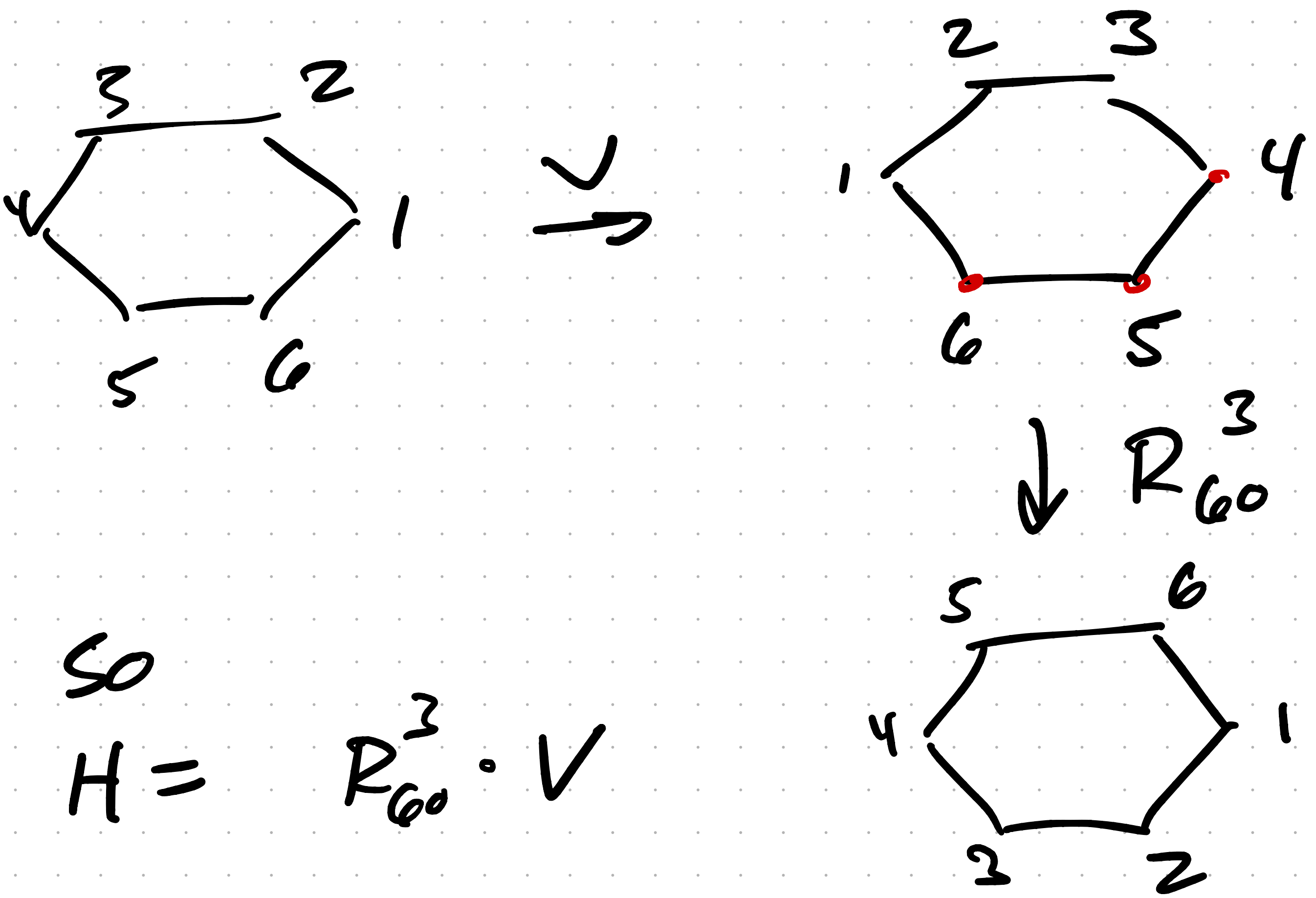
& extend h across words

so $h(aba^2) = R_{60^\circ} V R_{60^\circ}^2 \in D_6$

This is surjective b/c every element of D_6 is a combination of R_{60° and V

Ex





So

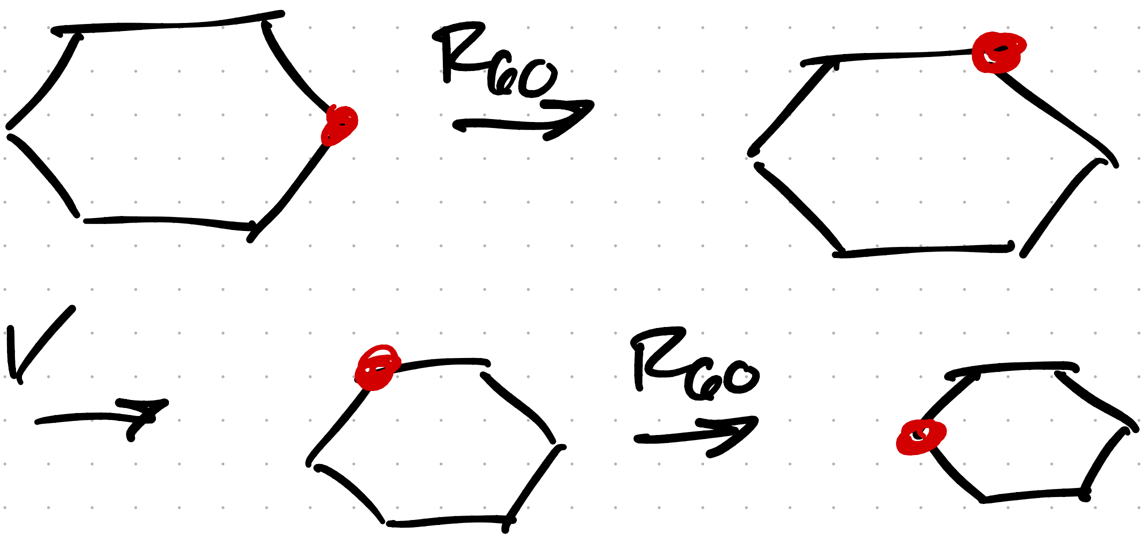
$$H = R^3_{60} \cdot V$$

Notice that

$$h(a) = R_{60} \qquad h(b) = V$$

$$h(a^6) = R_{60}^6 = \mathbb{1} \qquad h(b^2) = V^2 = \mathbb{1}$$

$$h(baba) = \underbrace{V R_{60} V}_{\text{rotation}} R_{60}$$



$$\Rightarrow \quad V R_{60} V R_{60} = \mathbb{1}$$

FACT

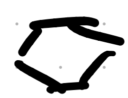
If w is a word in $\{a, b, a^{-1}, b^{-1}\}$

~~then~~ and if

$h(w) = \mathbb{1} \in D_6$

then w is the combination of

aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$, $\underline{a^6}$, $\underline{b^2}$, $\underline{baba}$

Upshot $D_6 = \mathrm{Isom}($  $)$

is generated by a, b $(= R_{60}, V)$

& has relations $a^6, b^2, baba$

GROUP PRESENTATIONS

A = alphabet ex $\{a, b\}$

$\{$words in $A \cup A^{-1}\}$ Ex $ab^2a^{-1}b^3$

$R \subset \{$words$\}$ Ex a^6, b^2
$baba$

relations

Define group:

elements are words w/ cancellations allowed & operation is concatenation

cancellations $\sim\sim a a^{-1} \sim\sim = \sim\sim \cancel{a}\cancel{a^{-1}} \sim$

$\sim\sim a^{-1} a \sim\sim = \sim\sim \cancel{a^{-1}}\cancel{a} \sim$

$\sim\sim b^{-1} b \sim\sim = \sim\sim \cancel{b^{-1}}\cancel{b} \sim$

$\sim\sim b b^{-1} \sim\sim = \sim\sim \cancel{b}\cancel{b^{-1}} \sim$

$\forall r \in R$

$\sim\sim \cancel{r} \sim\sim = \sim\sim \cancel{r} \sim\sim$

$\sim\sim \cancel{r^{-1}} \sim\sim = \sim\sim \cancel{r^{-1}} \sim\sim$

<u>Ex</u> $A = \{a, b\}$

$R = \{a^6, b^2, baba\}$

an example of an element of group is

$$a b^5 a^8 b a^7 b a b^3$$

$$= a \cancel{b^2} \cancel{b^2} b \cancel{a^6} a^2 b \cancel{a^6} a b a \cancel{b^2} b$$

$$= a b a^2 \cancel{baba} b$$

$$= a b a^2 b$$

Group is denoted $\langle A \mid R \rangle$

<u>Fact</u> Group

$\exists$ bijective homomorphism

$\langle a, b \mid a^6, b^2, baba \rangle$ is $\boxed{\text{isomorphic}}$

to D_6

We say that $\langle a, b \mid a^6, b^2, baba \rangle$ is a presentation for D_6