Groups from GT POV Pt 2

Ⓘ Classifying ISOMETRIES of $\mathbb{R}^2$

Ⓘⓘ Dihedral Groups

Ⓘ **Lemma** If $f \in \text{ISOM}(\mathbb{R}^2)$ and if

$$f(0,0) = (0,0)$$
$$f(1,0) = (1,0)$$
$$f(0,1) = (0,1)$$

then $f = id$

Lemma If $f \in \mathrm{Isom}(\mathbb{R}^2)$

and if $f(0,0) = (0,0)$

$f(1,0) = (1,0)$

$f(0,1) = (0,1)$

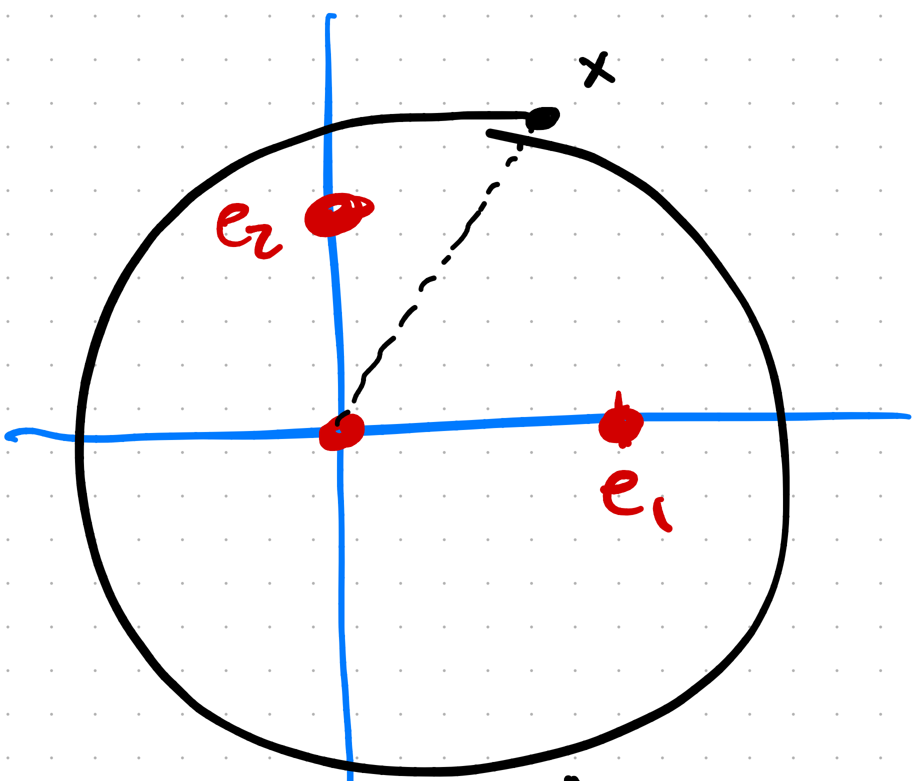
0

e_1

e_2

then $f = id$

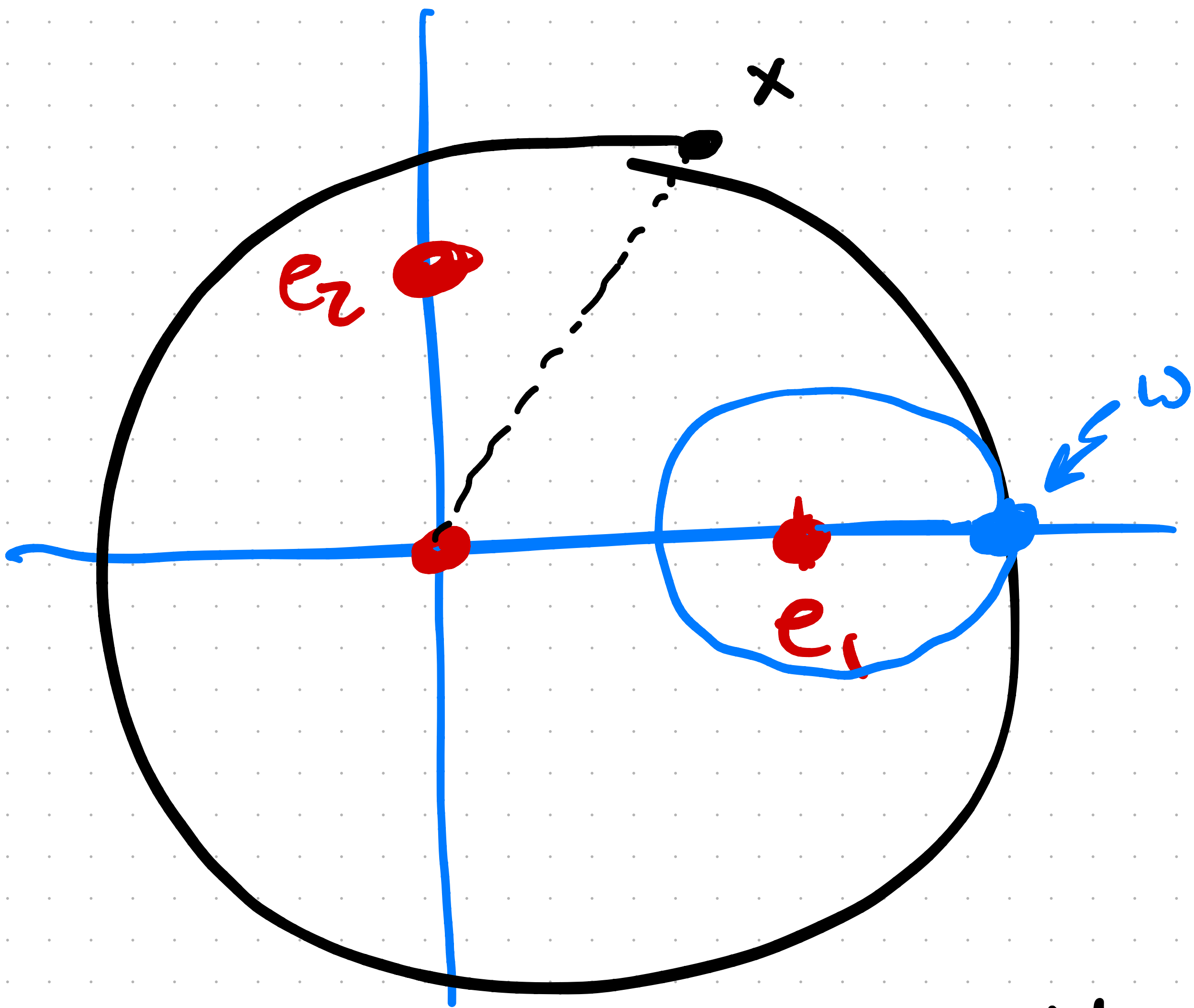
Show $f(x) = x$ $\forall x \in \mathbb{R}^2$



$f(x)$ some where on circle

b/c $d(f(x), f(0)) = d(x, 0)$

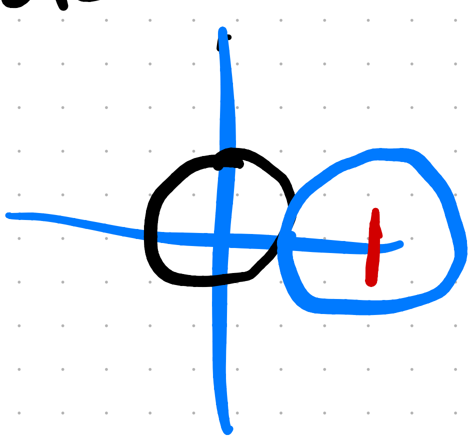
0

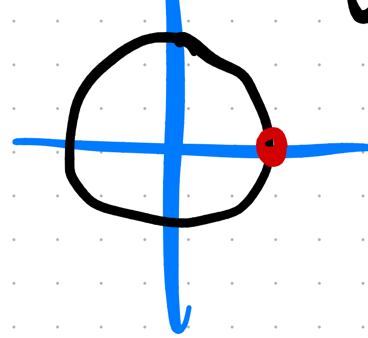


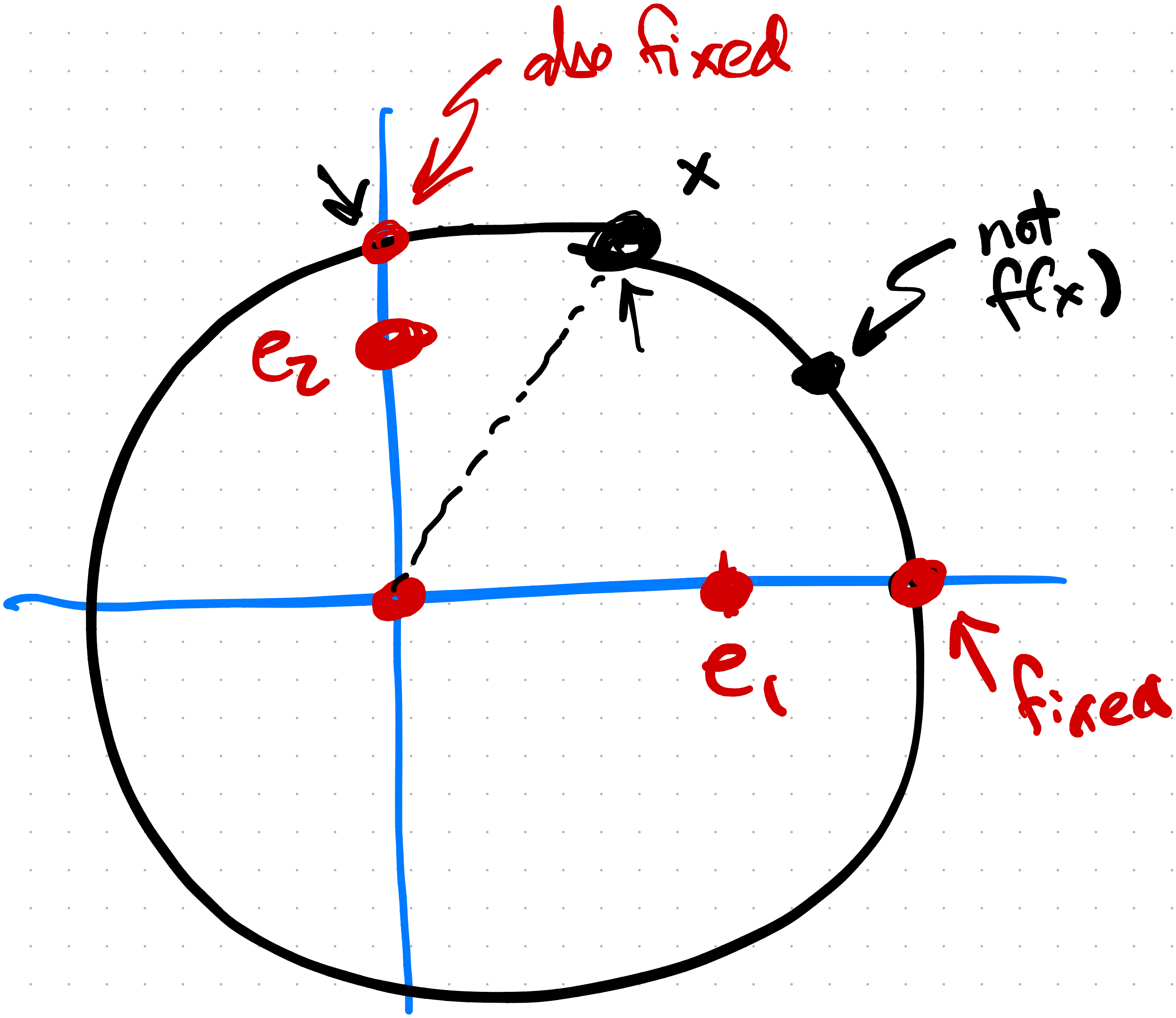
We know that $f(w)$ is on blue circle b/c dist from e_1 doesn't change & on the black circle

$\Rightarrow$ w is also fixed by f

Note we could have

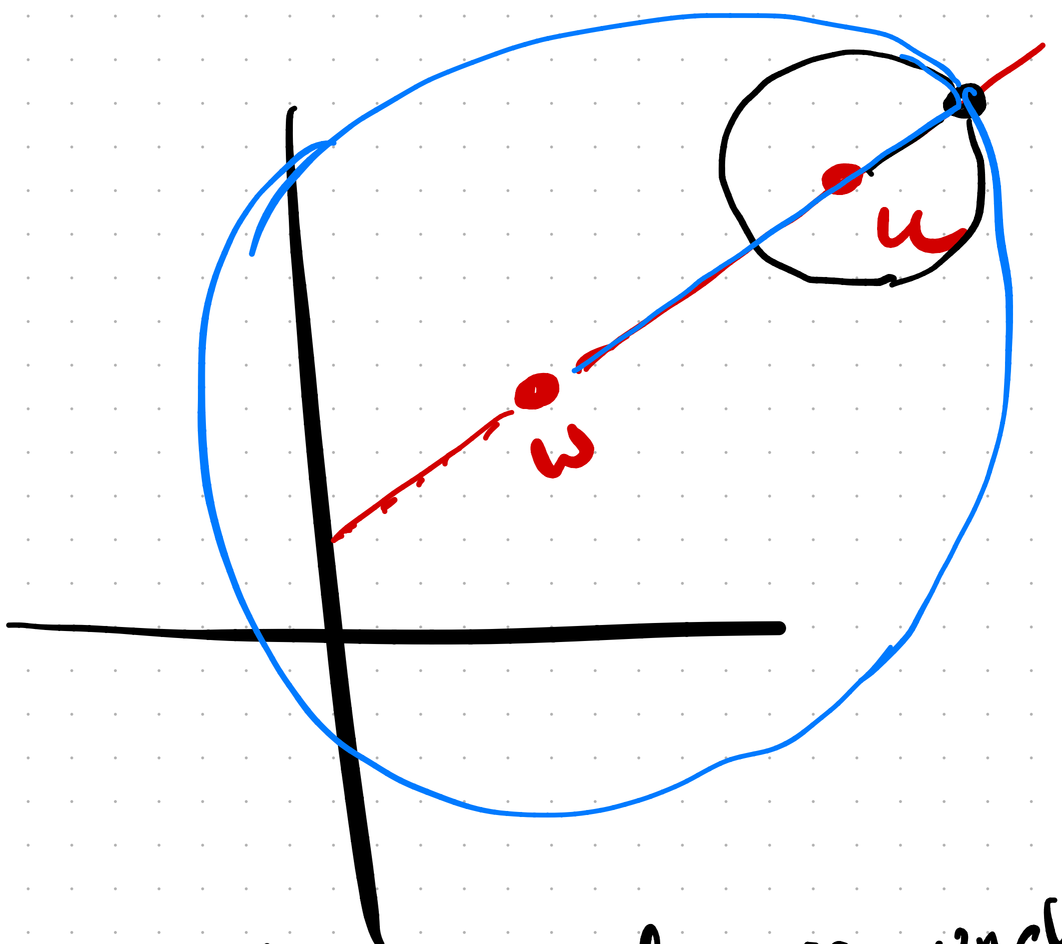






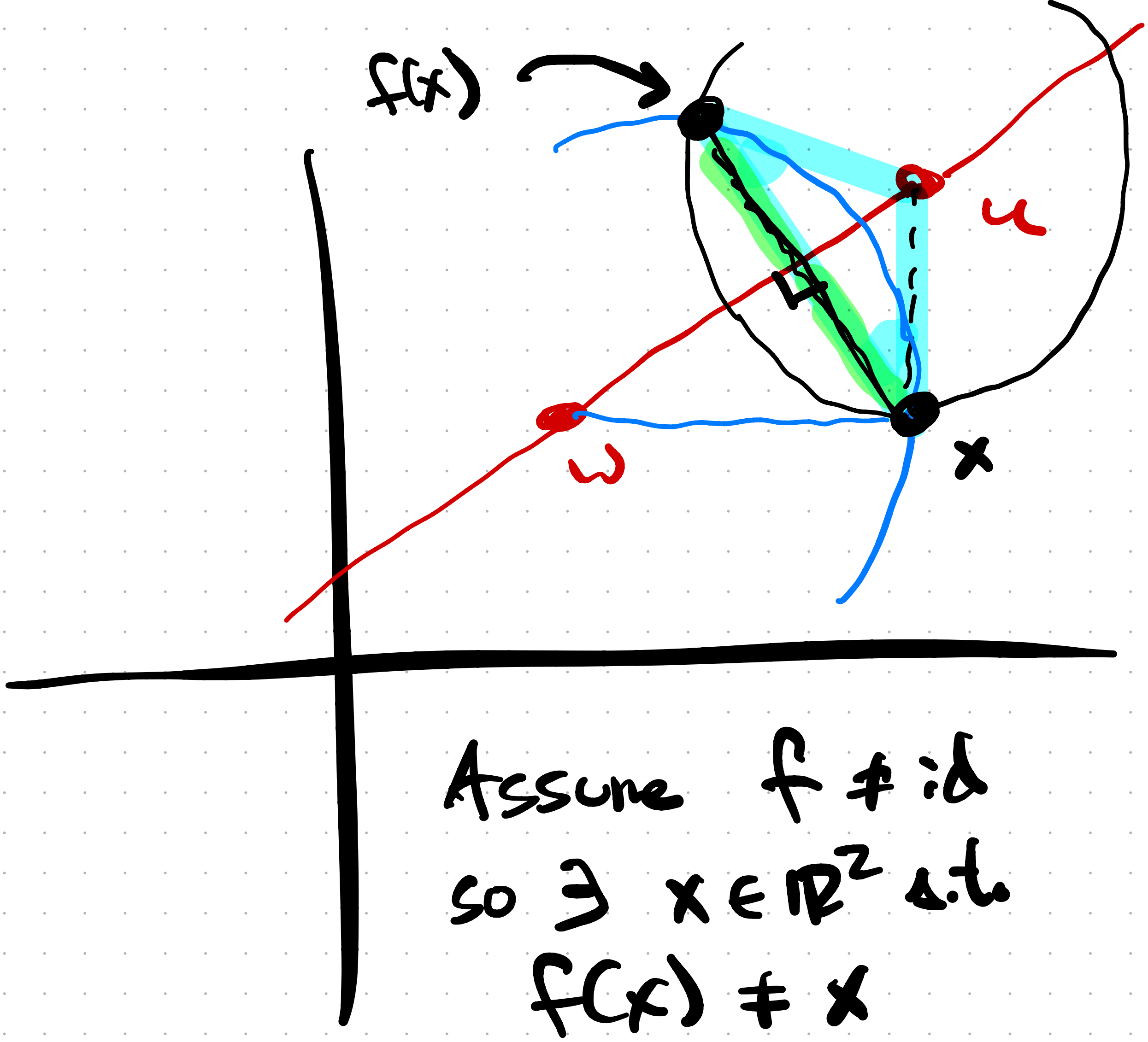
By looking distances to e_1 and e_2 we see that f preserves the quadrants of the black circle & so $f(x) = x$ □

Lemma If $f \in \mathrm{Isom}(\mathbb{R}^2)$ fixes two distinct pts then either f is the identity or f is the reflection across the line through those pts



Assume
$f(w) = w$
$f(u) = u$

B/c dist. to u and w is unchanged & blue & black circles are tangent
$\Rightarrow f$ fixes each pt on the line.

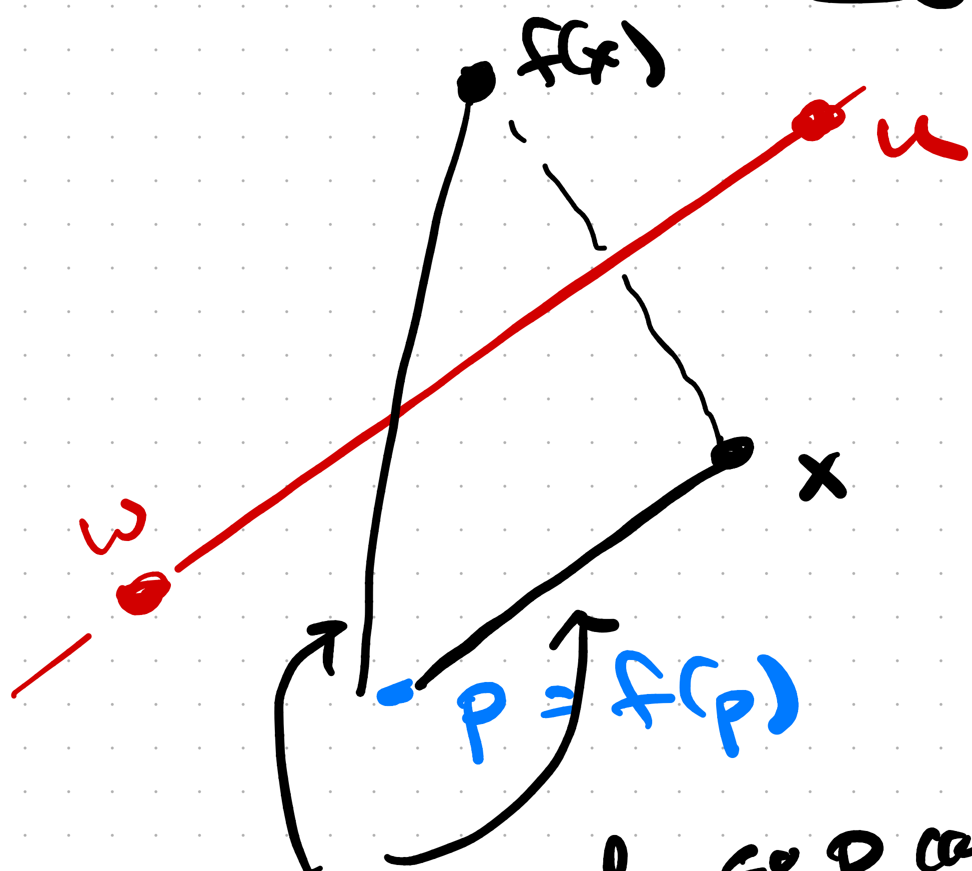


Assume $f \neq id$
so $\exists\ x \in \mathbb{R}^2$ s.t.
$f(x) \neq x$

By looking the circle & using facts about isos. triangle we $f(x)$ is the reflection of x across red line

We know now that if $f(x) \neq x$ then $f(x)$ is the refl. of x across line through w and u

We want this for <u>every</u> x.



not equal so p can't exist.

$\square$

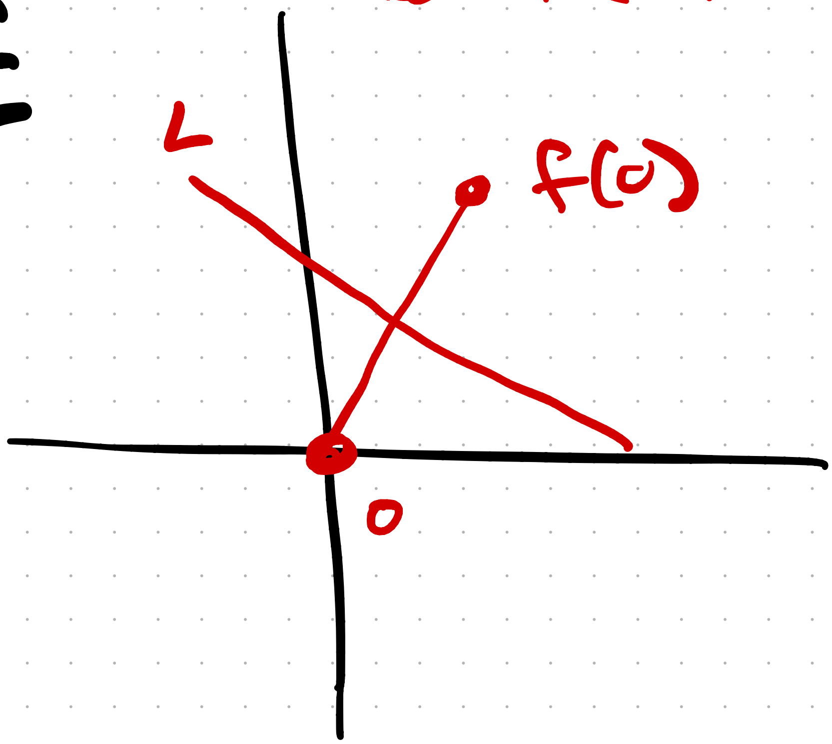
Lemma If $f \in \text{Isom}(\mathbb{R}^2)$ and if $\exists\, w \in \mathbb{R}^2$ s.t. $f(w) = w$ then either f is a rotation around w (possibly id) OR f is a reflection.

(HW)

* **Thm** If $f \in Isom(\mathbb{R}^2)$ then f is the composition of rotations, reflections

Let $f \in Isom(\mathbb{R}^2)$

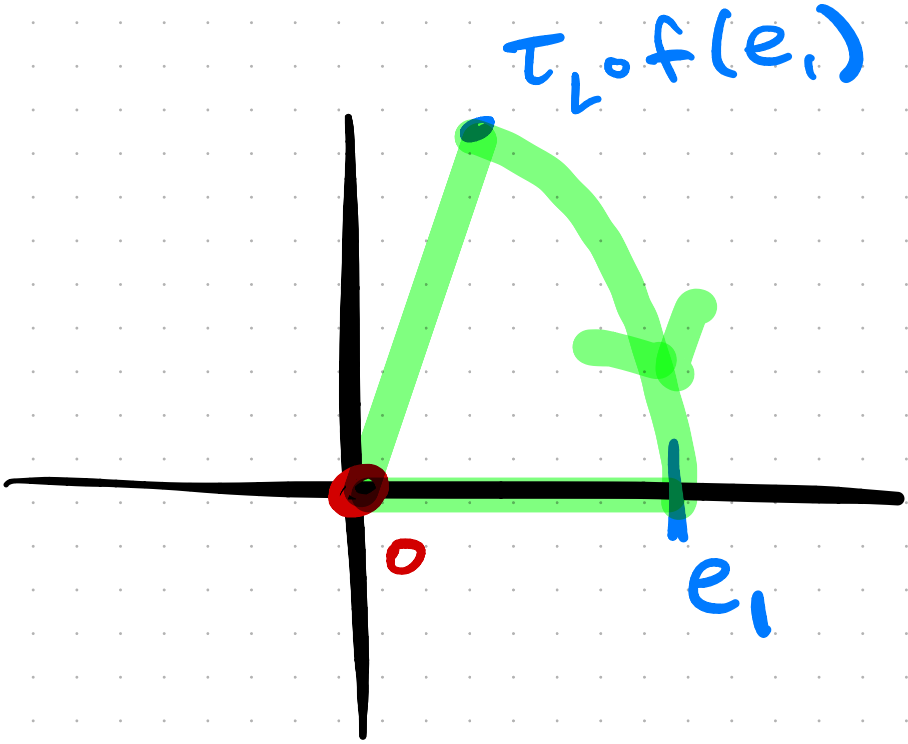
Pf



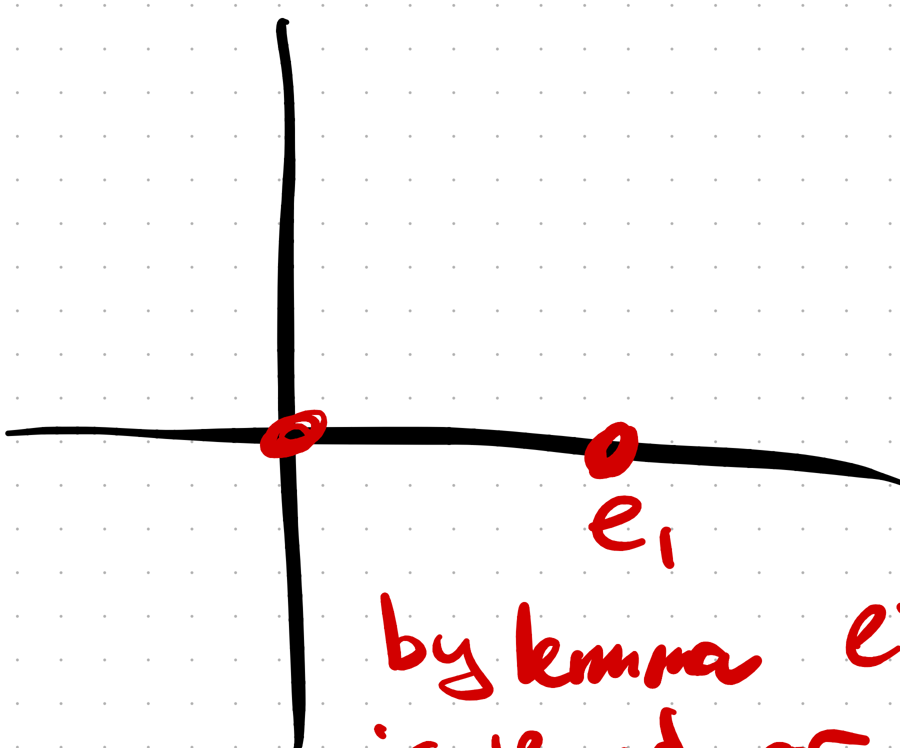
If $f(0) \neq 0$ let τ_L be the reflection across L

If $f(0) = 0$ let $\tau_L = id$

Then $\tau_L \circ f(0) = 0$



Let R be the rotation taking $\tau_L \circ f(e_1)$ to e_1 and fixing 0



$R \circ \tau_L \circ f$ fixes 0 and e_1

by lemma either $R \circ \tau_L$ is the id. or reflection across x-axis

If $R \circ \tau_L \circ f$ is the id

then $R \circ \tau_L \circ f(x) = x \quad \forall x \in \mathbb{R}^2$

$\Rightarrow f(x) = \tau_L^{-1} \circ R^{-1}(x)$

rotation!

either id or τ_L

in fact $\tau_L^{-1} = \tau_L$

$\Rightarrow$ f is the composition of a rotation & reflection

If $R \circ \tau_L \circ f$ = Refl across x-axis σ

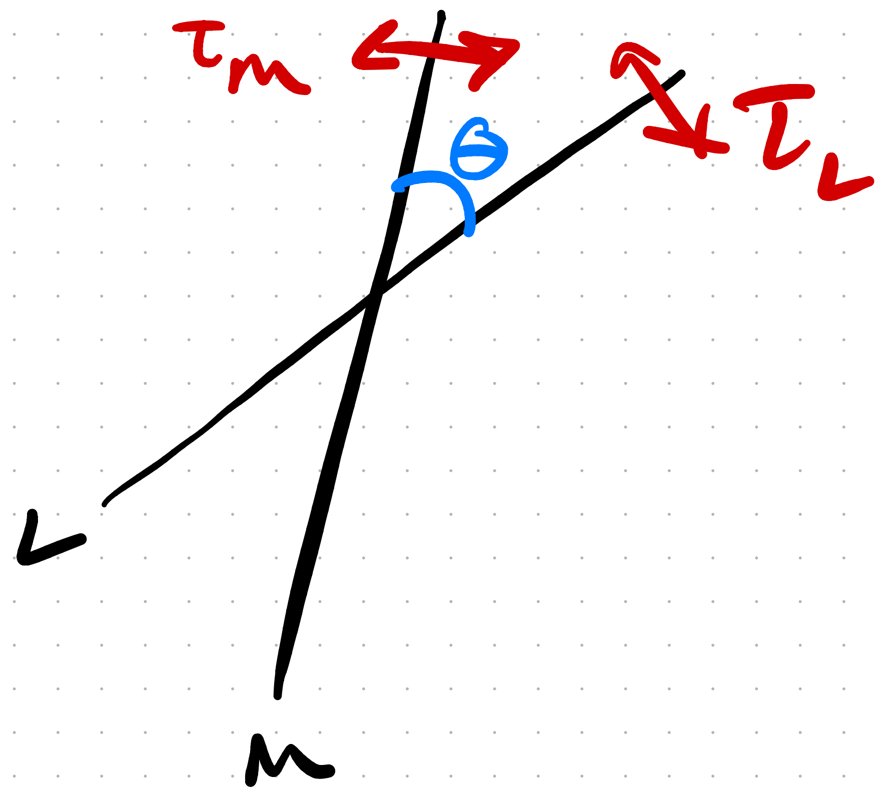
then $f = \tau_L \circ R^{-1} \circ \sigma$

rotation

reflection or id

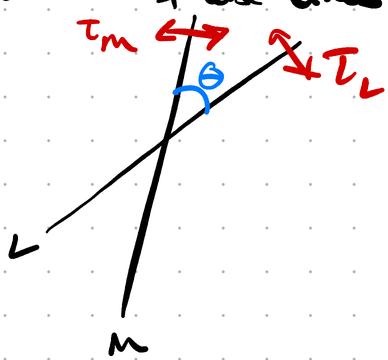
$\square$

Lemma If L & M are distinct lines intersecting at a pt P & if τ_L, τ_M are reflections across those lines



then $\tau_M \circ \tau_L$ is a rotation of angle 2θ

Lemma If L & M are distinct lines intersecting at a pt P & if τ_L, τ_M are reflections across those lines



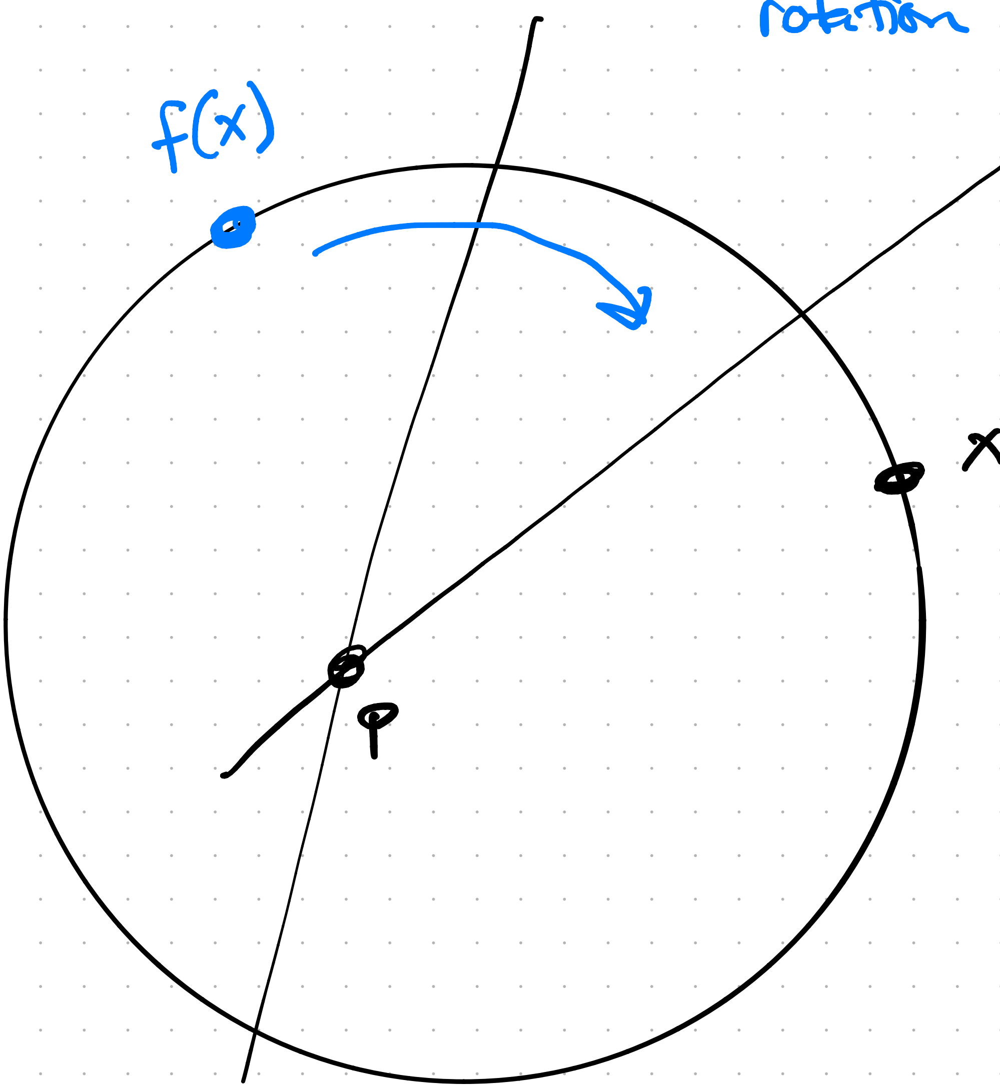
then $\tau_M \circ \tau_L$ is a rotation of angle 2θ

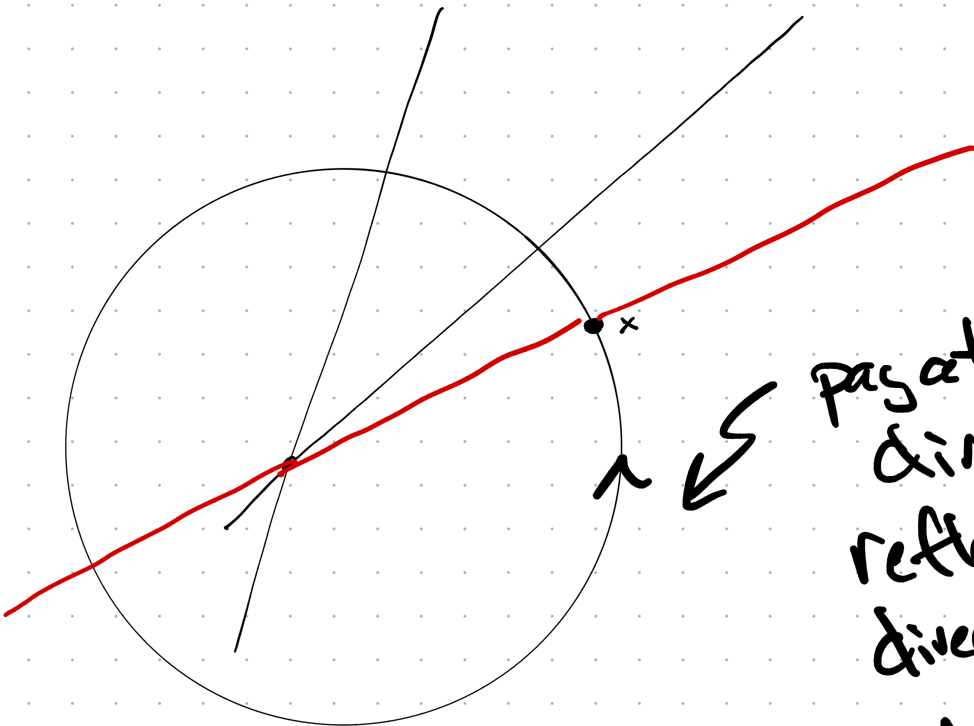
Pf

$\equiv$

$\tau_M \circ \tau_L$ preserves each circle centered at P

let R be the rotation taking $f(x)$ back to x





pay attention to direction

reflections reverse direction but rotations preserve direction

Know $R \circ \tau_M \circ \tau_L (x) = x$

and $R \circ \tau_m \circ \tau_L (0) = 0$

by lemma $R \circ \tau_m \circ \tau_L$ is either a reflection or ~~the~~ id.

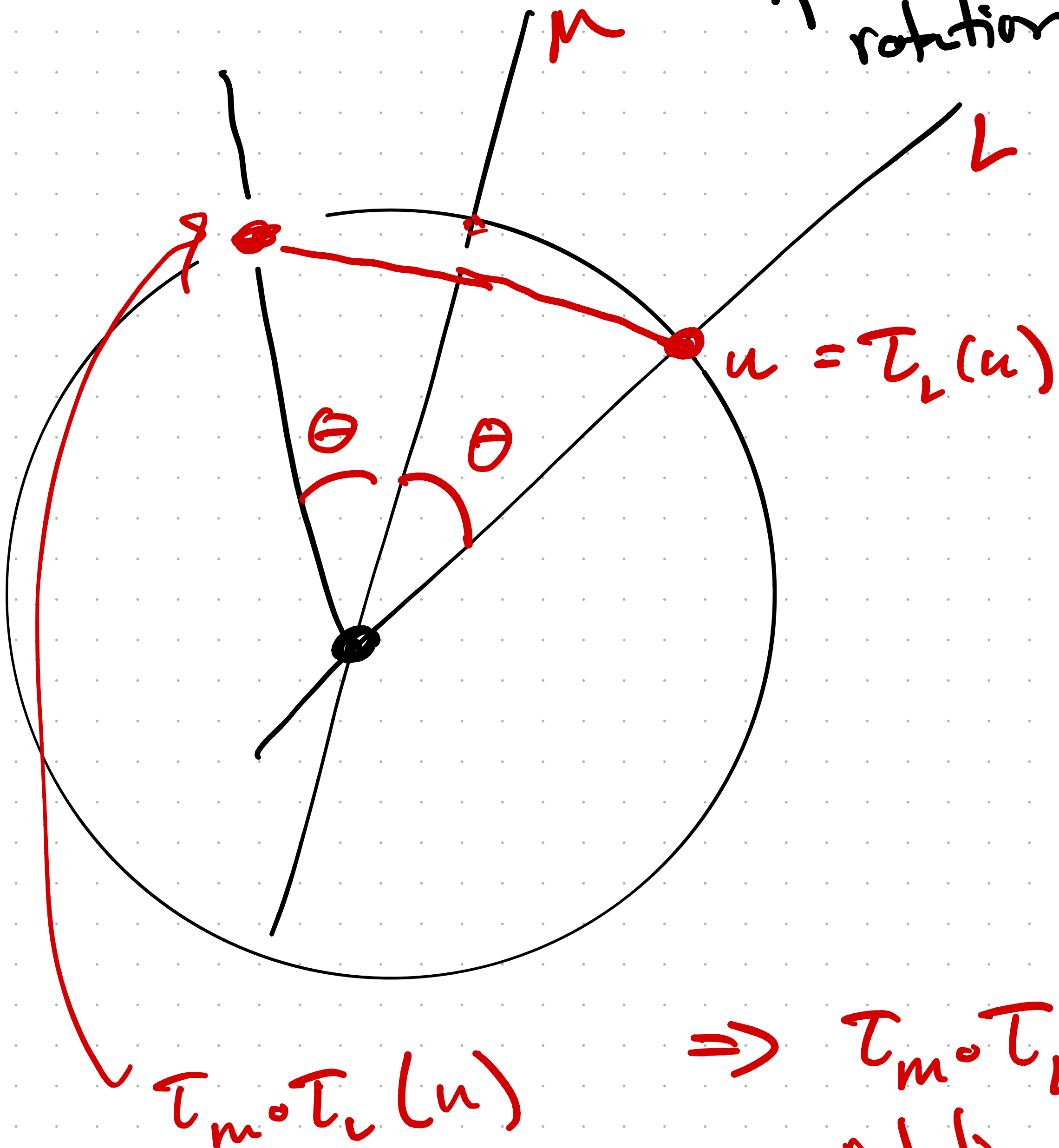
ea. reverse

$\tau_m \circ \tau_L$ preserves direction

$\Rightarrow$ $R \circ \tau_m \circ \tau_L$ preserves direction

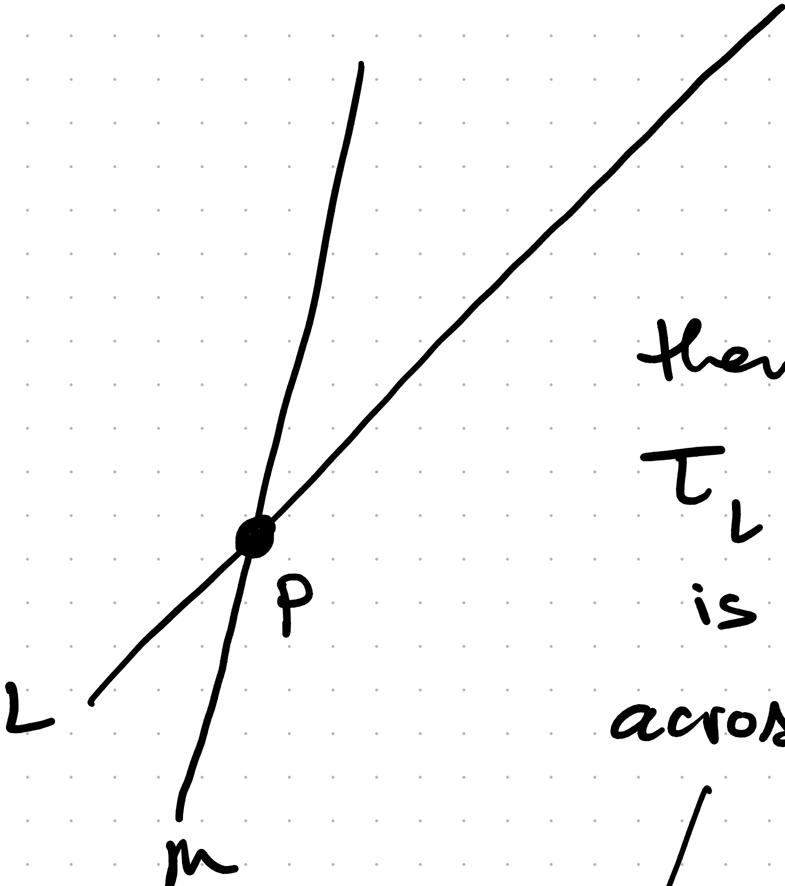
$\Rightarrow$ $R \circ \tau_m \circ \tau_L = id$

$$\tau_m \circ \tau_L = \underset{\text{rotation}}{R^{-1}}$$



$\Rightarrow$ $\tau_m \circ \tau_L$ is a rotation by angle 2θ

<u>Thm</u> Let L, M be distinct lines intersecting at P

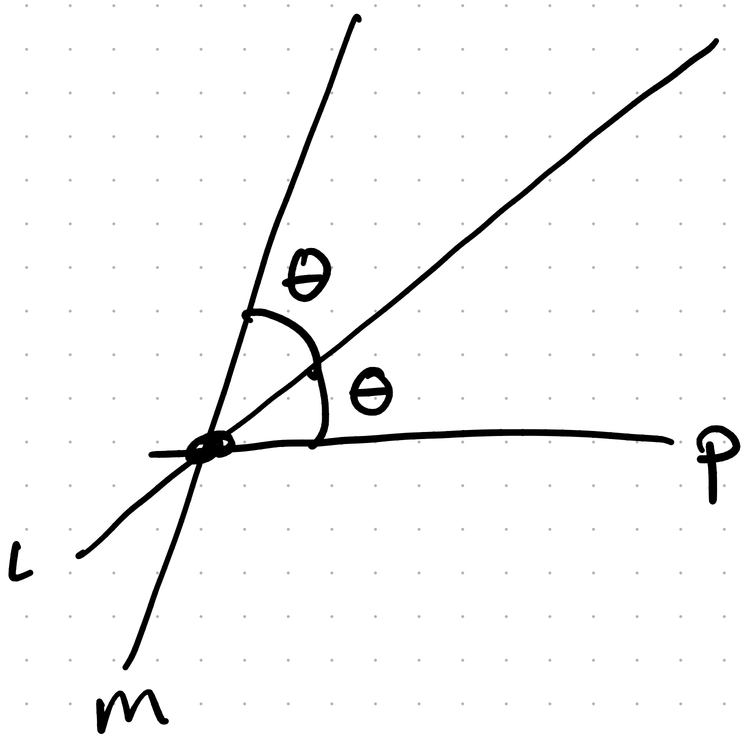


Let τ_L & τ_M be the reflections

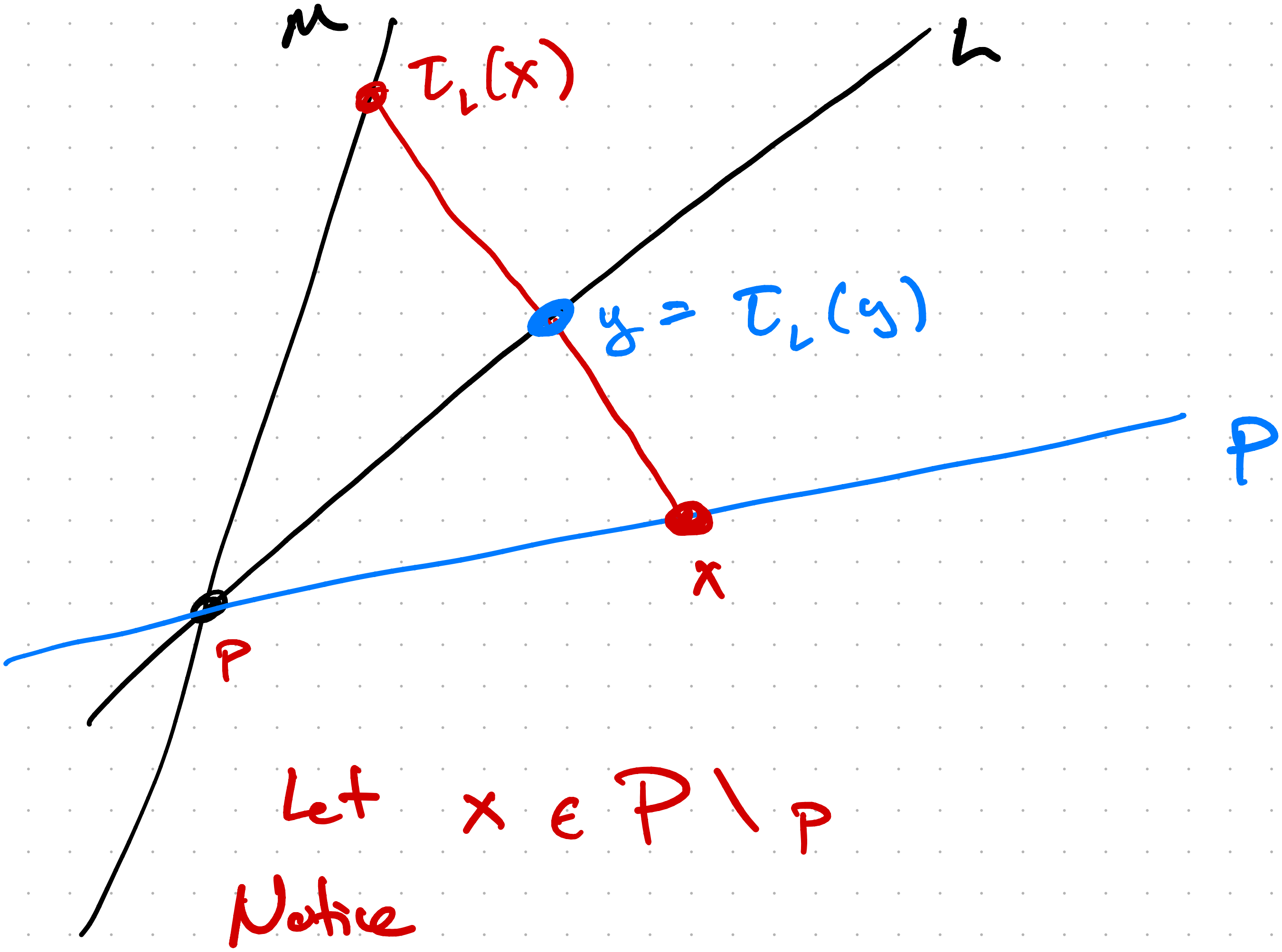
then

$\tau_L \circ \tau_m \circ \tau_L$

is a reflection across the line P



Pf



Let $x \in P \setminus p$

Notice

$$\tau_L(x) \in M$$

and $\tau_M \circ \tau_L(x) = \tau_L(x)$

and $\tau_L \circ \tau_M \circ \tau_L(x) = x$.

so $\tau_L \circ \tau_M \circ \tau_L$ is the ~~id~~ or reflection across P.

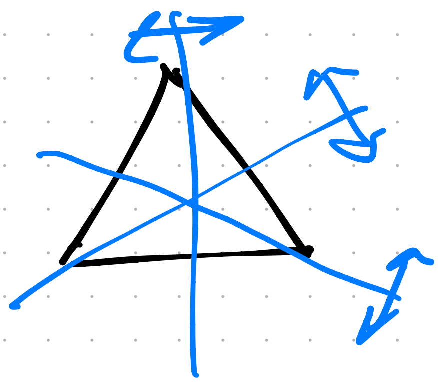
bc pts of L are moved

$\square$

II Dihedral Groups

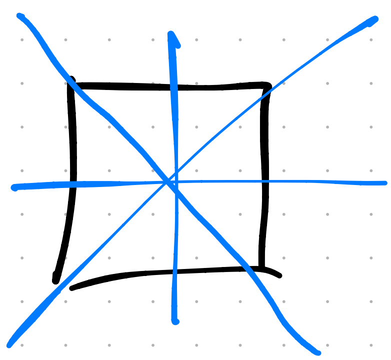
The dihedral group D_n is the isometry group of a regular n-gon

<u>Ex</u> D_3 = symmetries of $\triangle$



3 rotations
(by $0°$, $120°$, $240°$)
3 reflections

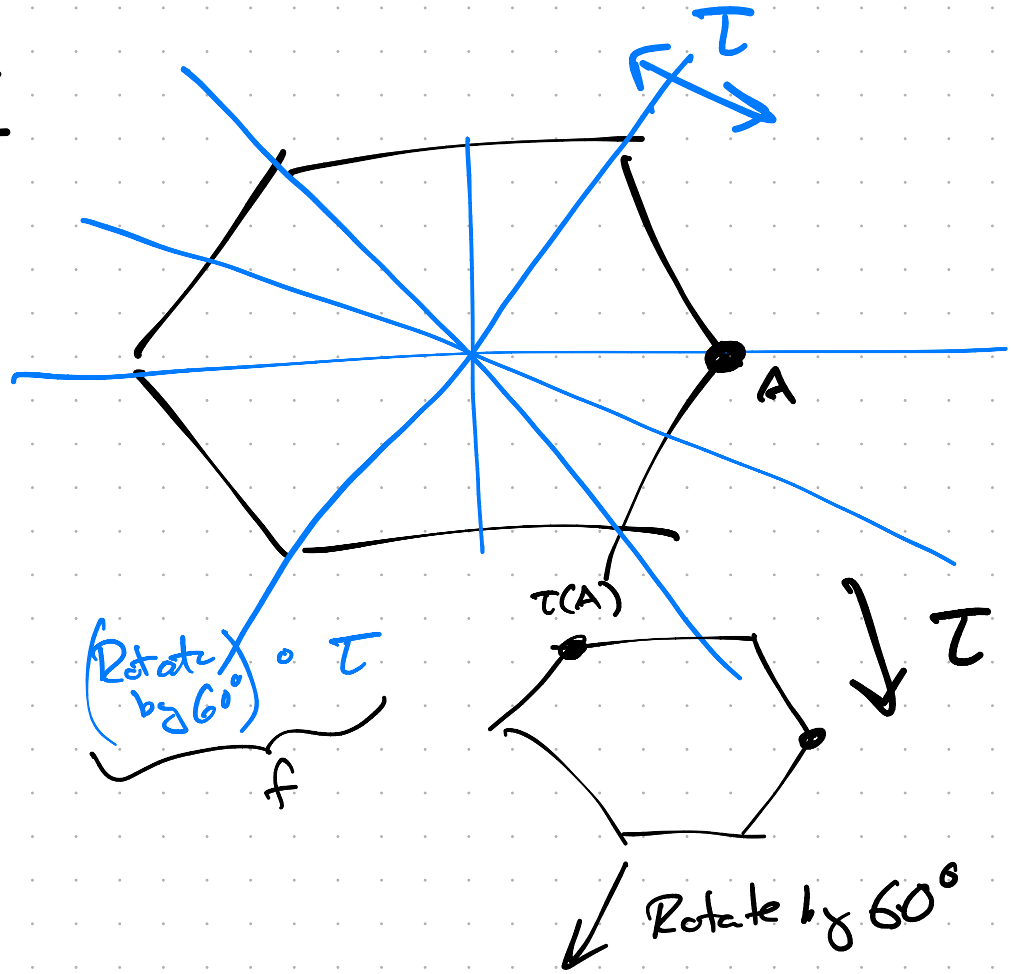
<u>Ex</u>

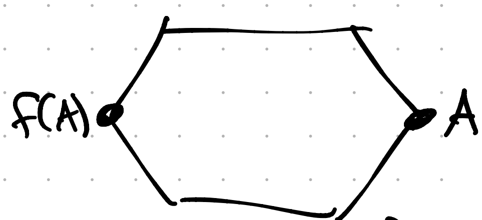


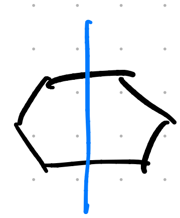
4 rotations

In general, D_n has n rotations, n reflections

Ex





⇒ f is either rotation by 180° or 

It must be the reflection

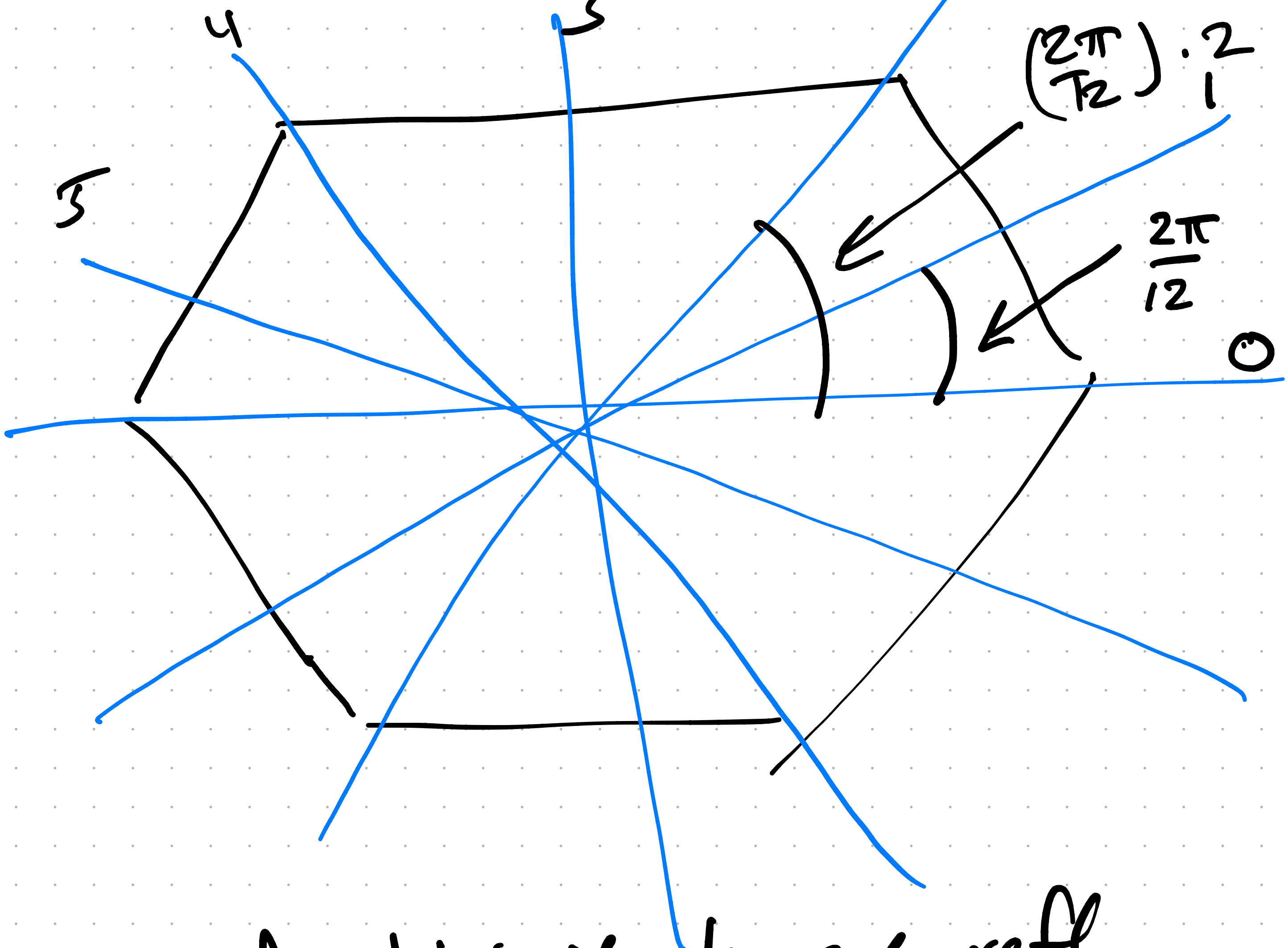
b/c f reverses orientation

(reflection τ ↺ ⇝ ↻

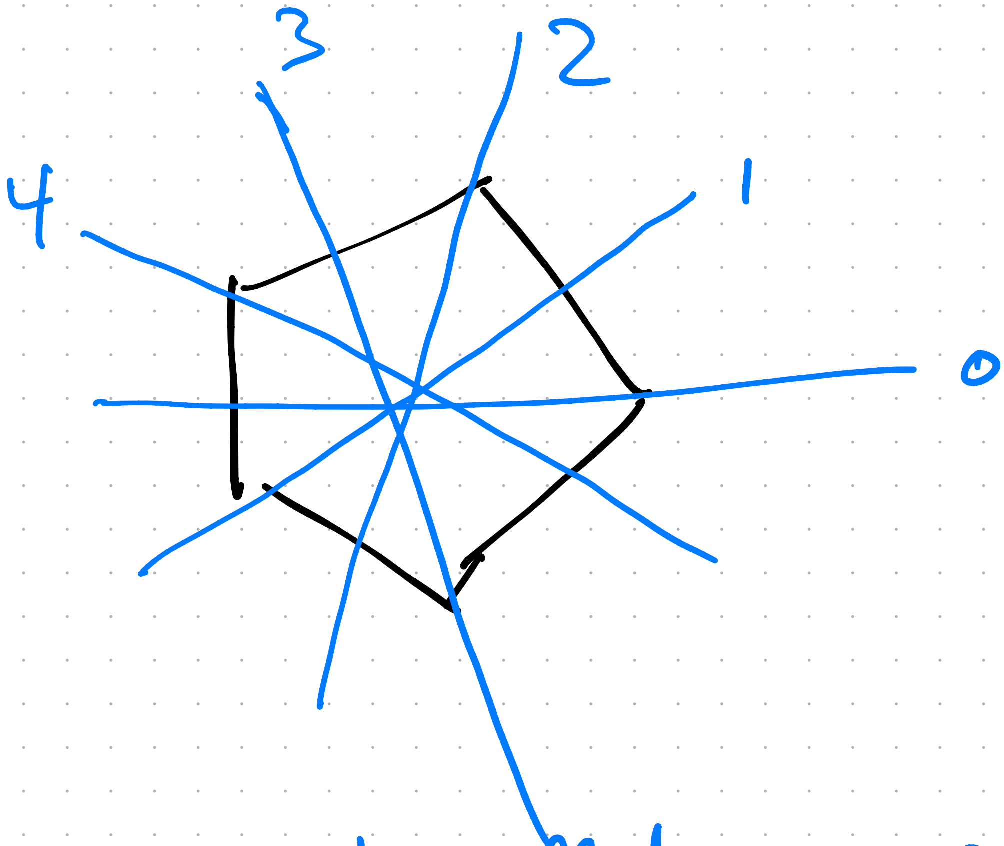
rotation ↻ ⇝ ↻

□

Consider refl. lines

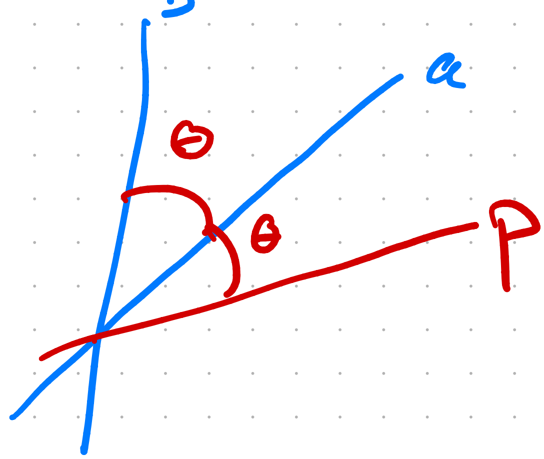


In general label the lines of refl. of the n-gon $0, \dots, n-1$ so the line at angle $\left(\frac{2\pi}{2n}\right)j$ has label j



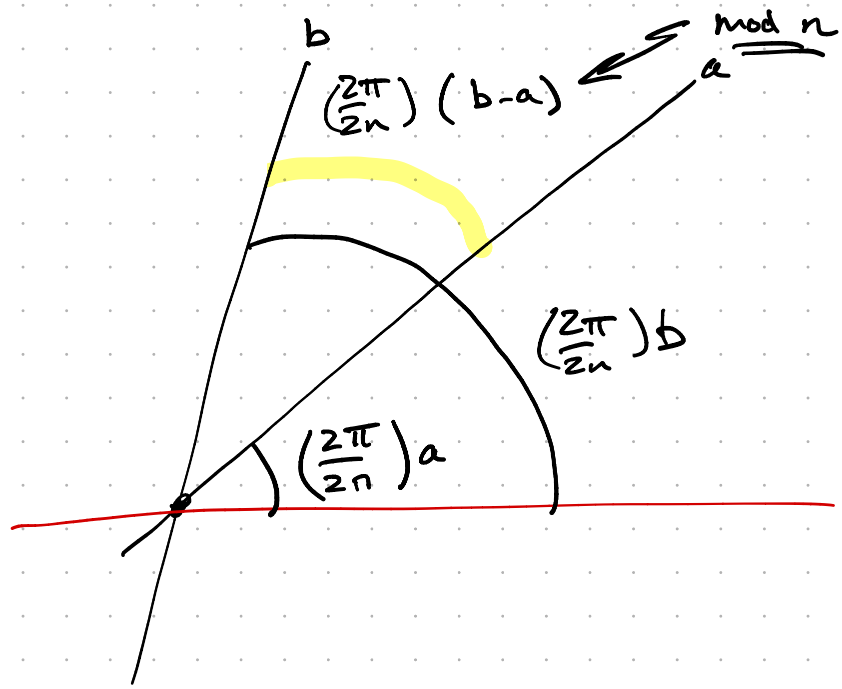
Let τ_i be reflection across line i

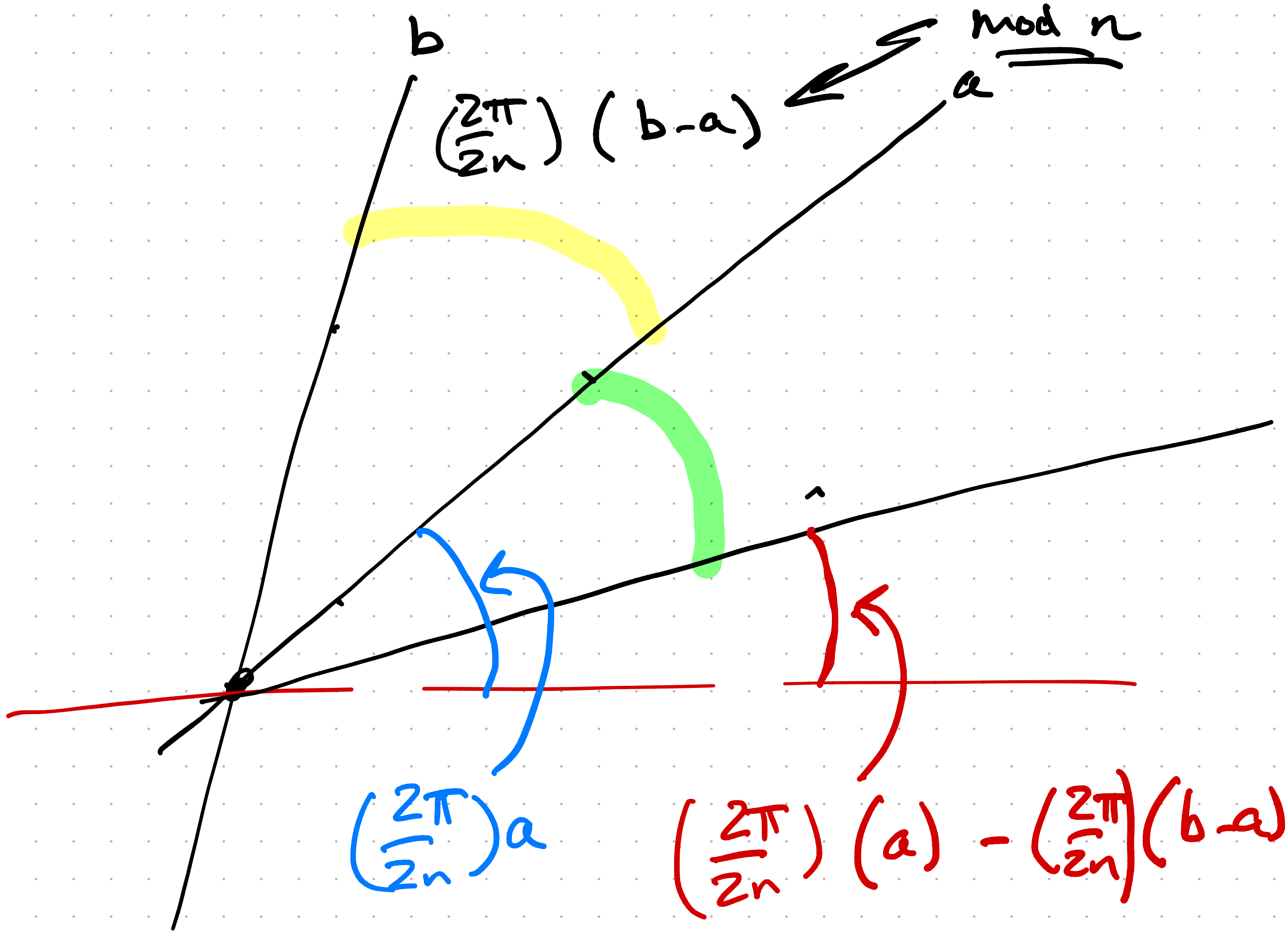
consider



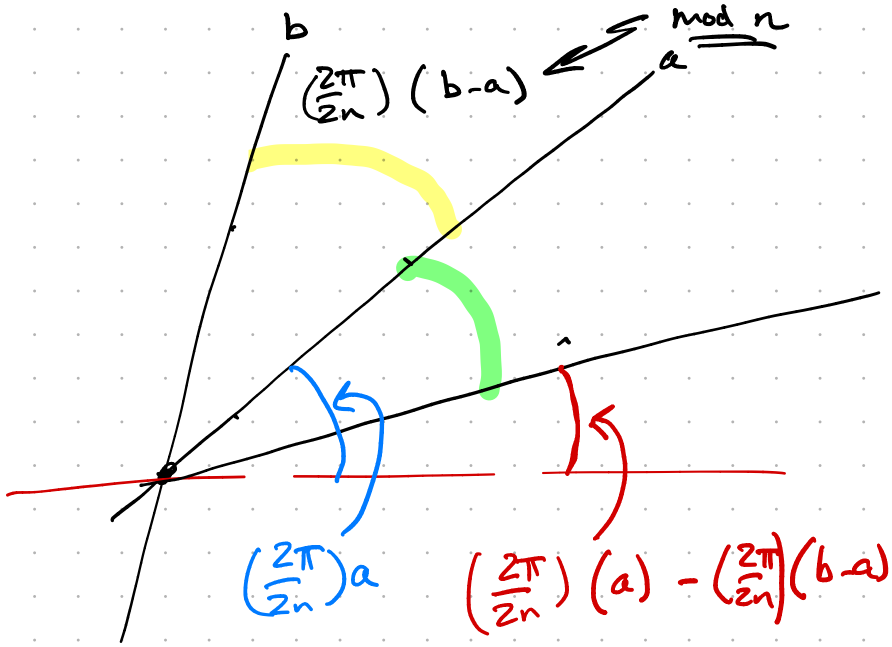
$\tau_a \circ \tau_b \circ \tau_a$

know this is the reflection across P





$$= \left(\frac{2\pi}{2n}\right)(2a-b)$$



$$= \left(\frac{2\pi}{2n}\right)(2a-b)$$

We conclude that

$$\tau_a \circ \tau_b \circ \tau_a = \tau_{\underline{2a-b}}$$

Q: What's the connection between dihedral groups & knot coloring?