Groups from Geometry / Top. POV

Def A group is a <u>set</u> G & a way of combining elements of G, that is a function $G \times G \to G$

$$a \quad b \to ab$$

s.t.

① $\forall\, a, b \in G \quad ab \in G$

② (Identity) $\exists\ \mathbb{1} \in G$ s.t. $\forall\, a \in G \quad \mathbb{1} \cdot a = a\mathbb{1} = a$

③ (Inverses) $\forall\, a \in G \ \exists\, a^{-1} \in G$ s.t $aa^{-1} = a^{-1}a = \mathbb{1}$

④ (Associative) $\forall\, a, b, c \in G$

$$a(bc) = (ab)c = abc$$

Usually $ab \neq ba$ (usually <u>not</u> commutative)

EXAMPLES

① $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

operation $+$

- If $a, b \in \mathbb{Z}$ $a+b \in \mathbb{Z}$
- $\forall a \in \mathbb{Z}$ $a+0 = 0+a = a$

  so 0 is the identity

- $\left.\begin{array}{l} a+(-a) = 0 \\ (-a)+a = 0 \end{array}\right\} \forall a \in \mathbb{Z}$

  so every $a \in \mathbb{Z}$ has an inverse $-a$.

- Assoc. $\forall a, b, c \in \mathbb{Z}$

  $a+(b+c) = (a+b)+c$.

  Inherited Fact.

(2) Multiplication

$G = \mathbb{R}$ (real #s)

mult. $\cdot$ is the operation.

→ $\forall a, b \in \mathbb{R}$ $\quad a \cdot b \in \mathbb{R}$

→ $\forall a \in \mathbb{R}$ $\quad 1 \cdot a = a \cdot 1 = a$

→ $\forall a \in \mathbb{R}$

$$a \cdot \left(\frac{1}{a}\right) = \left(\frac{1}{a}\right) \cdot a = 1$$

What about $a = 0$??

0 does not have a multiplicative inverse so $\mathbb{R}$ w/ $\cdot$ is <u>not</u> a group.

(3) $\mathbb{R} \setminus \{0\}$ mult. $\cdot$ as the operation <u>is</u> a group.

3 $\mathbb{Z}/k\mathbb{Z}$ modular arithmetic
is a group w/ operation $+$

4 $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$

$$= \left\{ (a,b) \;\middle|\; \begin{array}{l} a \in \mathbb{Z}/5\mathbb{Z} \,\& \\ b \in \mathbb{Z}/7\mathbb{Z} \end{array} \right\}$$

group operation is $+$

$$(a,b) + (a',b') = (\underbrace{a+a'}_{\text{mod } 5}, \underbrace{b+b'}_{\text{mod } 7})$$

(5) X is a set

$\mathrm{Perm}(X) = \{ f : X \to X \mid f \text{ is a bijection} \}$

Operation is function composition $\circ$

→ Recall if $f, g : X \to X$ are bijections then $f \circ g : X \to X$ is a bijection

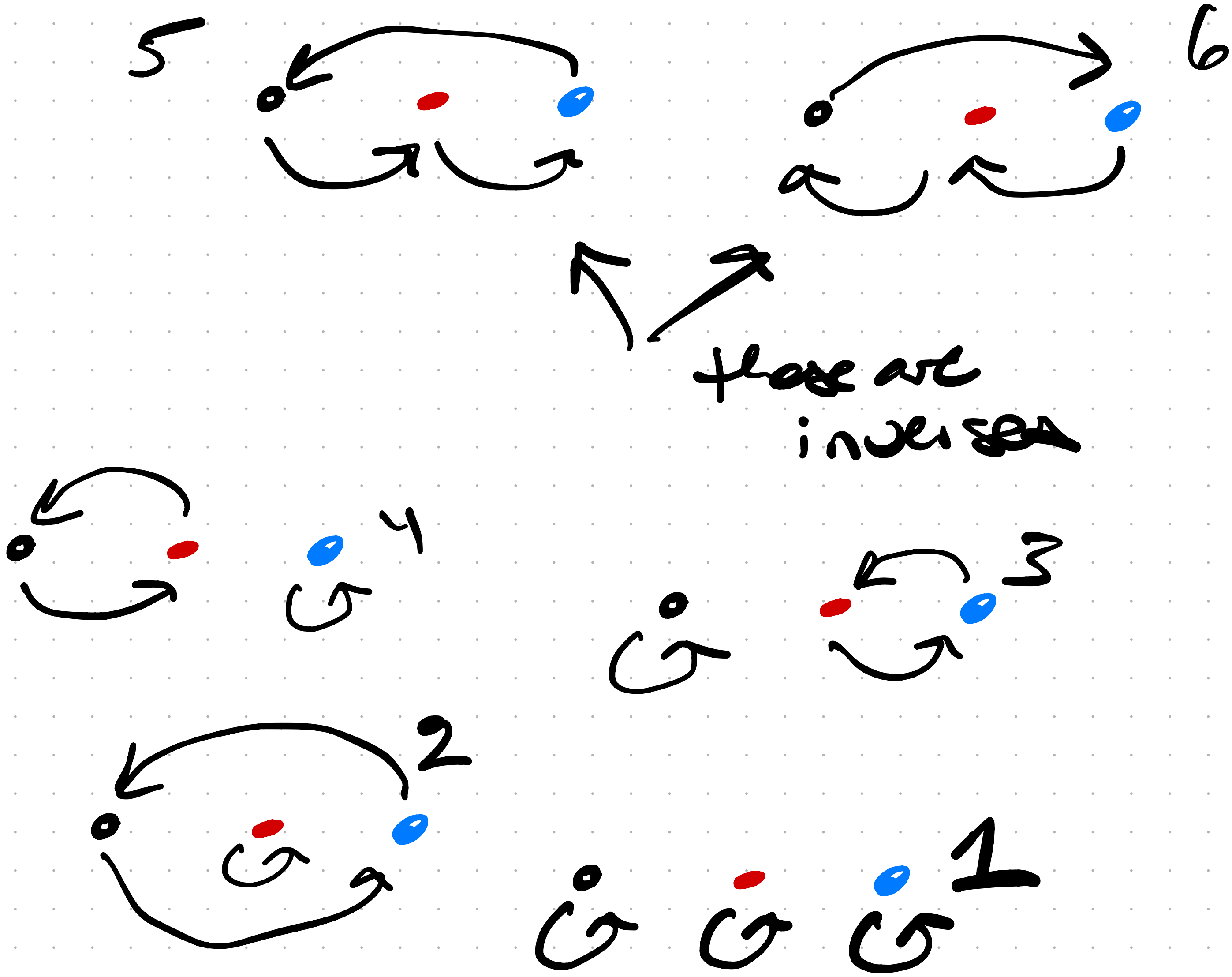
→ The identity is $id : X \to X$ defined by $id(x) = x \ \forall x \in X$

→ Inverse

→ Associative

Ex

$X = \{ \bullet, \bullet, \bullet \}$



Ex $GL_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc \\ \neq 0 \end{array} \right\}$

identity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

is $\dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

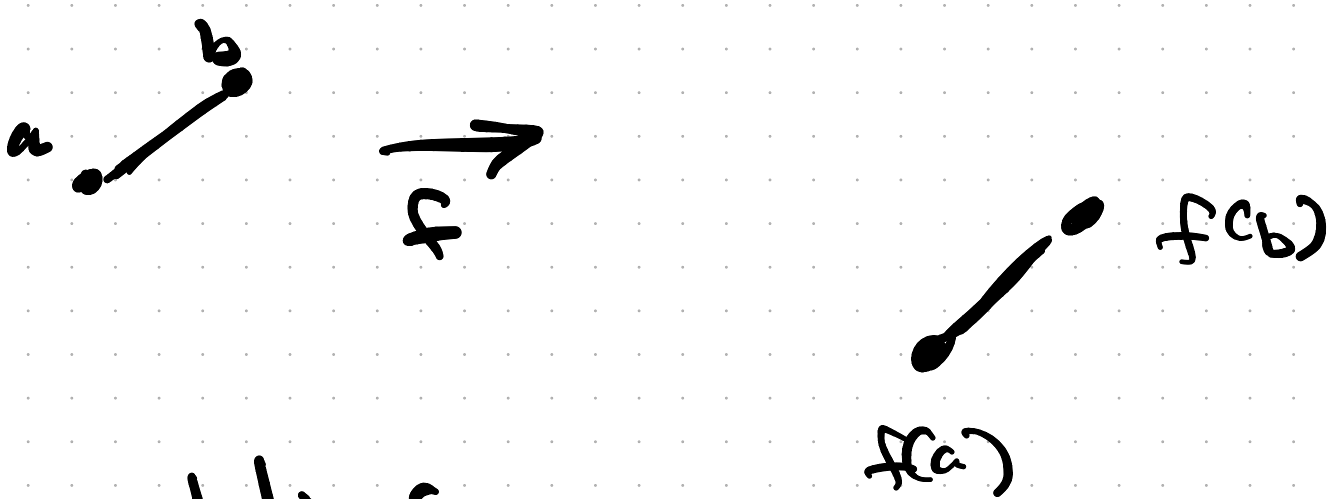
⑥ Isom$(\mathbb{R}^2)$

$= \{ f : \mathbb{R}^2 \to \mathbb{R}^2 \mid$ f is a bijection and $\forall\, a, b \in \mathbb{R}^2$

$d(a,b) = d(f(a), f(b))$

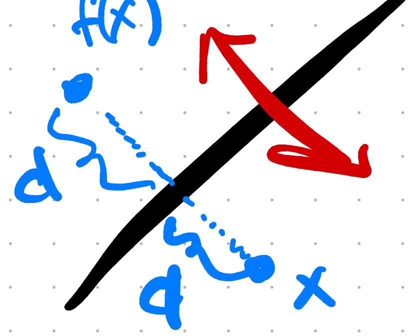
Ex translation

$f(x,y) = (x+7, y-\pi)$



rotations

reflections



Isom($\mathbb{R}^2$) is a group

the operation is composition

$\Rightarrow$ Associative

let's suppose $g, f \in \text{Isom}(\mathbb{R}^2)$

$a, b \in \mathbb{R}^2$

b/c g is an isom

$$d(a,b) = d(g(a), g(b))$$

$$= d(f(g(a)), f(g(b)))$$

$$= d(f \circ g(a), f \circ g(b))$$

b/c f is an isom

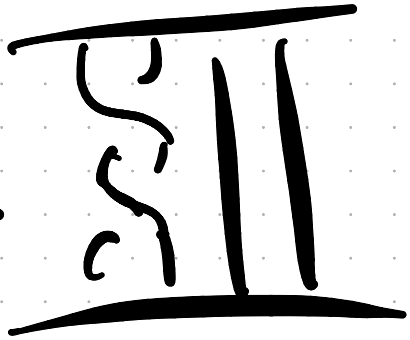
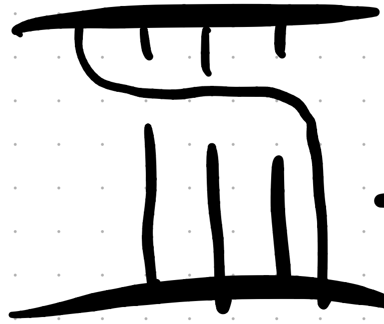
$$d(a,b) = d(\overbrace{f(a)}^{x}, \overbrace{f(b)}^{y}) \quad \forall x,y \in \mathbb{R}^2$$

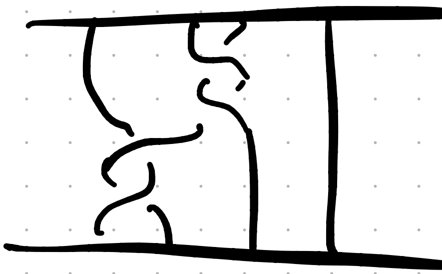
$$\Updownarrow$$

$$d(f^{-1}(x), f^{-1}(y)) = d(x,y)$$

7 Braid group B_n on n-strands

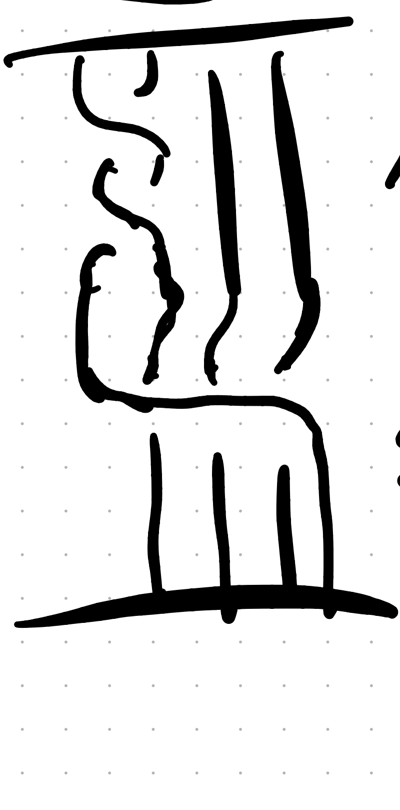
$\underline{n = 4}$

A   B



Q: WHAT IS THE INVERSE?

operation: stacking



A

AB

B

8 Free groups

Alphabet $= \{a, b, c\} = A$

$\{a^{-1}, b^{-1}, c^{-1}\} = A^{-1}$ disjt

A **word** in $A \cup A^{-1}$ is a sequence of symbols

$$a\,b\,b\,c\,c^{-1}\,b^{-1}\,a\,a\,a\,b\,c^{-1}\,b\,a^{-1}\,a^{-1}$$

$$= a\,b^2\,c\,c^{-1}\,b^{-1}\,a^3\,b\,c^{-1}\,b\,a^{-2}$$

Let $\mathcal{W} = \{\text{words in } A \cup A^{-1}\}$

operation is concatenation

$$(abca^{-1})(bc^{-1}ba^3)$$

$$= a\,b\,c\,a^{-1}\,b\,c^{-1}\,b\,a^3$$

Two words are equivalent if ~~we~~ they are related by inserting and deleting

$$x x^{-1} \text{ or } x^{-1}x \quad \text{for } x \in A$$

The empty word is denoted $\mathbb{1}$.

Working w/ the quotient set (under the equiv. relation) we have a group F_n if $n = \#$ of el'ts of A

Ex $A = \{a, b, c\}$

$$\underbrace{ab\cancel{c}\cancel{c^{-1}}ba}_{\text{different words}} = \underbrace{ab^2a}_{} \text{ in } F_n$$

different words

but the same elt of F_n

$$(ab^5ac^{-3})(c^4a^{-1}b)$$
$$= a\,b^5a\,c^{-3}c^4a^{-1}b$$
$$= a\,b^5a\,c\,a^{-1}b$$

F_n is a group identity is $\mathbb{1}$

inverse of a word is the same letters in opp. order w/ inverses

$(abc)^{-1} = c^{-1}b^{-1}a^{-1}$ b/c

$$(abc)(c^{-1}b^{-1}a^{-1}) = \cancel{a}\cancel{b}\cancel{c}\cancel{c^{-1}}\cancel{b^{-1}}\cancel{a^{-1}}$$
$$= \mathbb{1}.$$