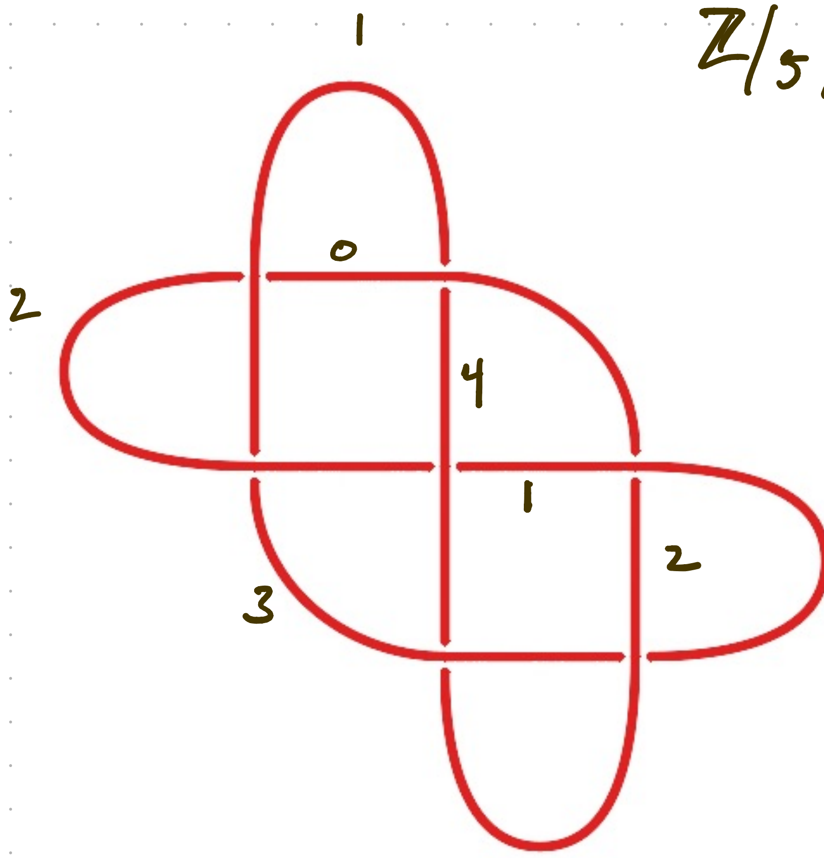
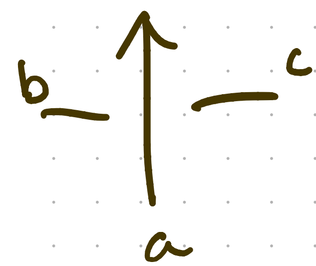
Coloring by groups



$\mathbb{Z}/5\mathbb{Z}$

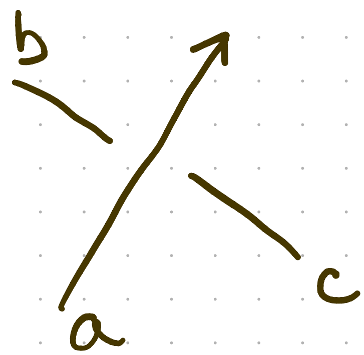


$$2a - b - c = 0$$

$$\underbrace{c = 2a - b}$$

reflections in dihedral group

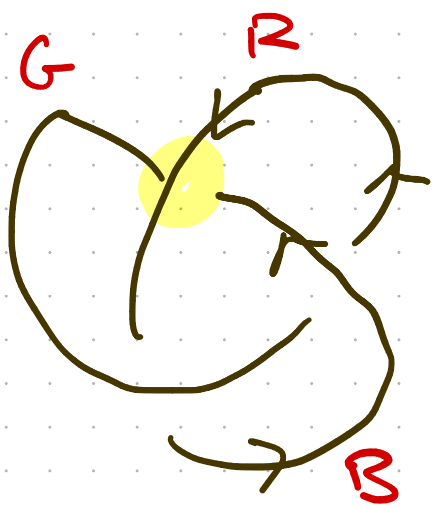
Def A coloring of a oriented diagram D by a group G is a function
$$\{\text{strands}\} \to G$$
s.t.



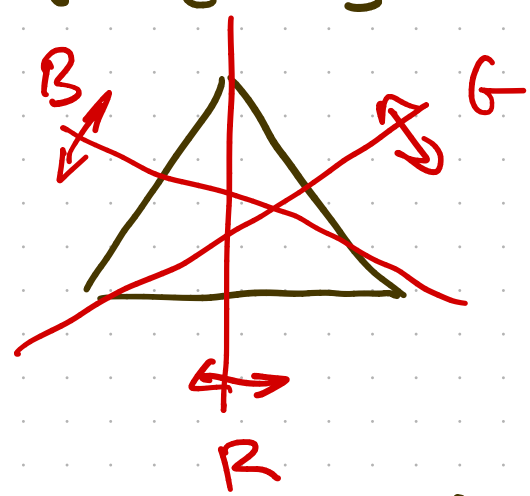
$$c = a b a^{-1}$$

$a, b, c \in G$

Ex



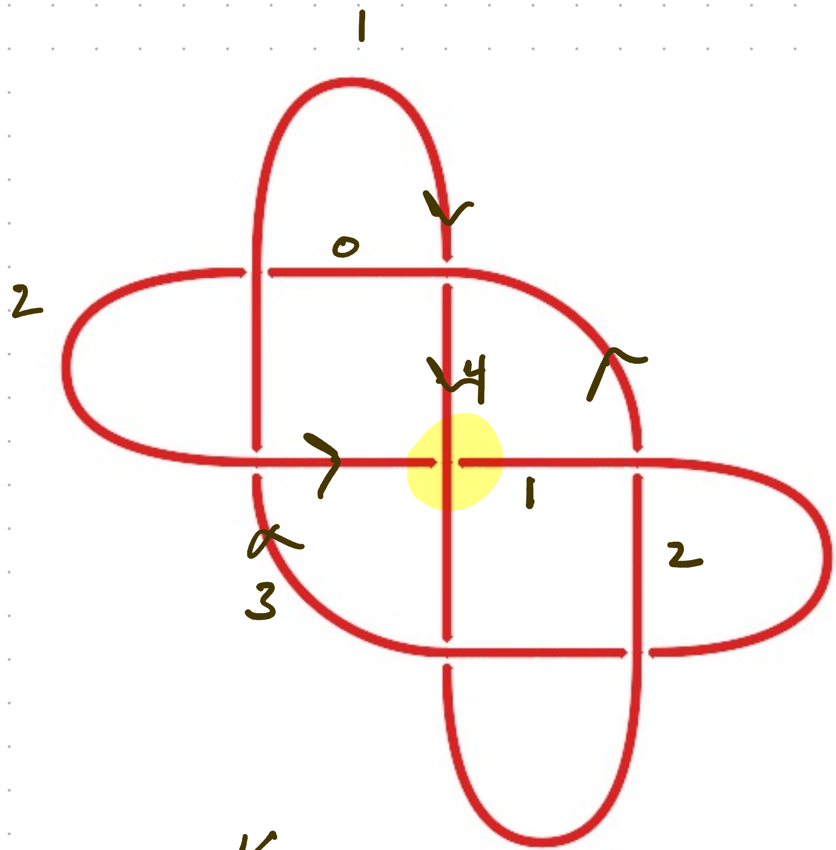
coloring by D_3



Need is $G = R B R^{-1}$ ← reflection $R^{-1} = R$

$$G = R B R$$

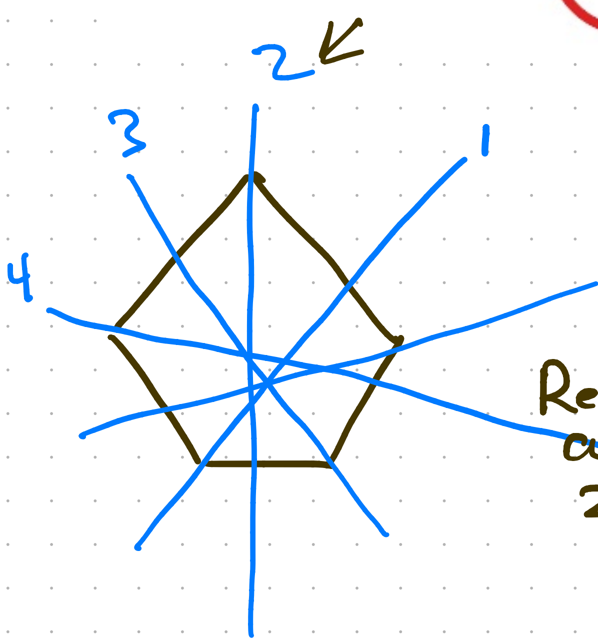
By work on dihedral groups: Yes!



$\mathbb{Z}/5\mathbb{Z}$

$\updownarrow$

reflections in D_5



To be a coloring by D_5

At we need

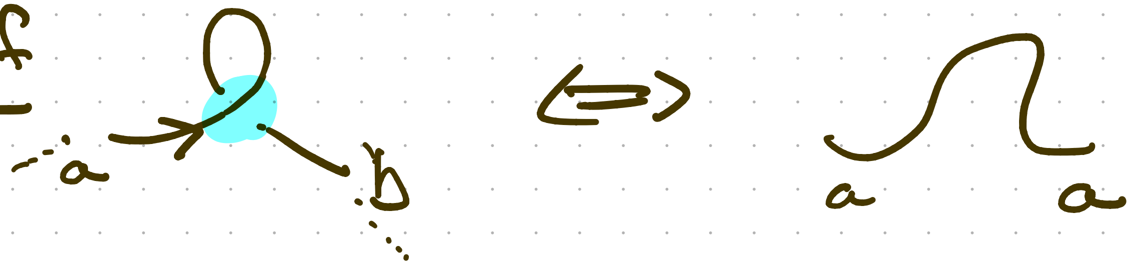
$$\text{Refl. across } 2 \stackrel{?}{=} \text{Refl across } 4 \circ \text{Reflect across } 1 \circ \text{Refl across } 4$$

$\Updownarrow$

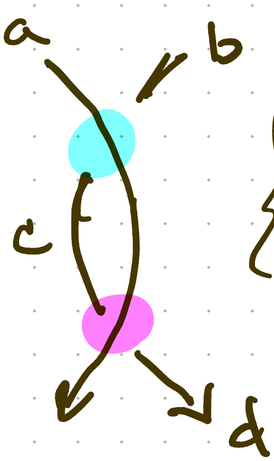
$$\begin{aligned} 2 &= 2 \cdot 4 - 1 \mod 5 \\ &= 8 - 1 \\ &= 7 \\ &= 2 \mod 5 \ \checkmark \end{aligned}$$

Thm If a diagram D' is obtained from a diagram D by a Reidemeister move & if D has a non-trivial coloring by G then so does D'.

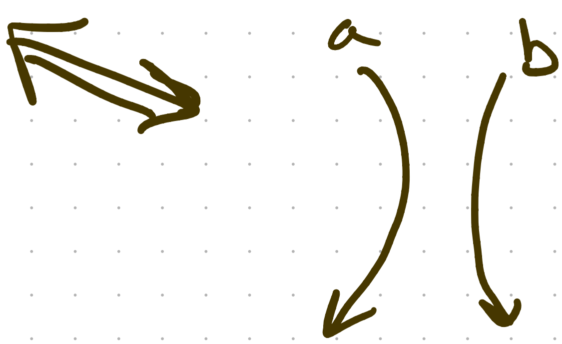
Pf

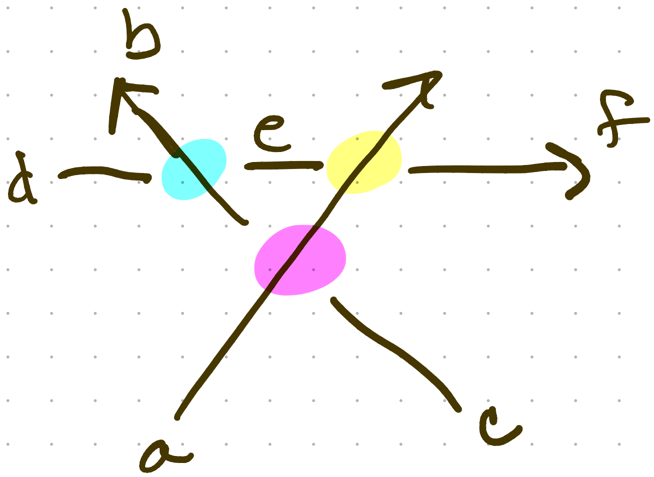


$$
\begin{aligned}
b &= a\,\cancel{a}\,\cancel{a}^{-1} \\
&= a
\end{aligned}
$$



$$
\begin{cases} c = a b a^{-1} \\ c = a d a^{-1} \end{cases} \iff \begin{cases} b = a^{-1} c a \\ d = a^{-1} c a \end{cases} \iff b = d
$$

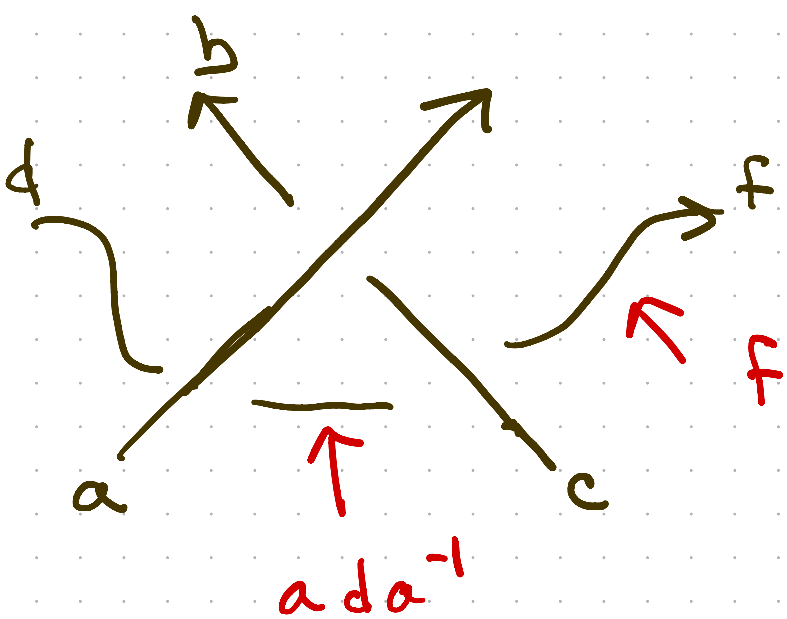




$c = aba^{-1}$

$e = bdb^{-1}$

$f = aea^{-1}$



$f = cada^{-1}c^{-1}$

From top eqns: $f = abdb^{-1}a^{-1}$

$b = a^{-1}ca$

$$\Downarrow$$

$$f = \cancel{a}\,\cancel{a^{-1}}\,c\,a\,d\,a^{-1}c^{-1}\,\cancel{a}\,\cancel{a^{-1}}$$

$$= cada^{-1}c^{-1}$$

✓

★ Also check other Reidemeister versions □

Note the connection w/ K-colorings

A diagram D is K-colored

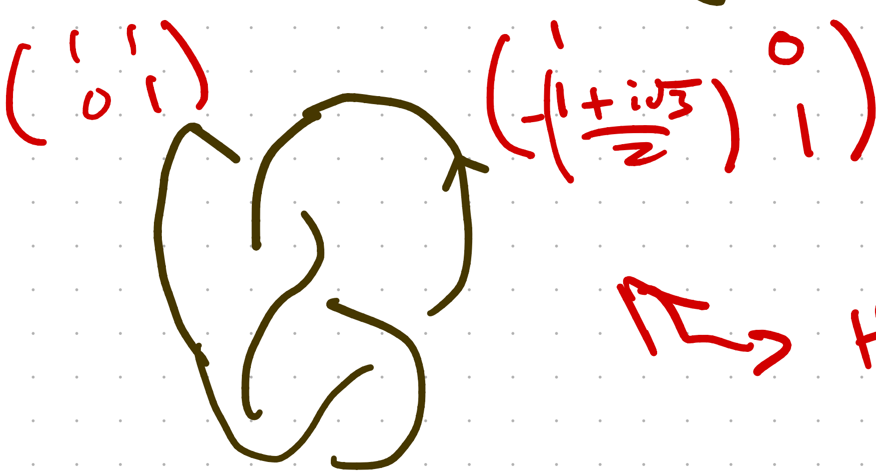
$\iff$ D is colored by D_K using reflections.

But using other groups we can get many more types of colorings!

<u>Ex</u> Use the group $PSL_2\mathbb{C}$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad-bc=1 \right\}$$

But declare $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$



$\rightsquigarrow$ Hyperbolic Geometry