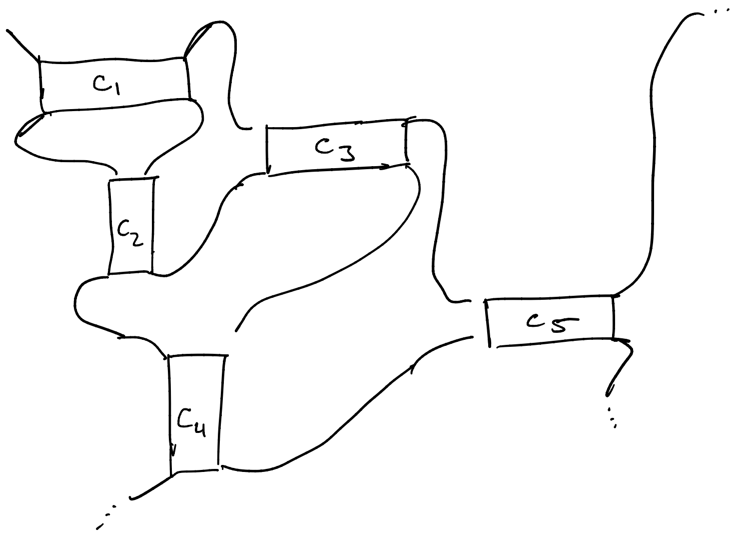
Conway's Theorem for Rational Tangles

based on

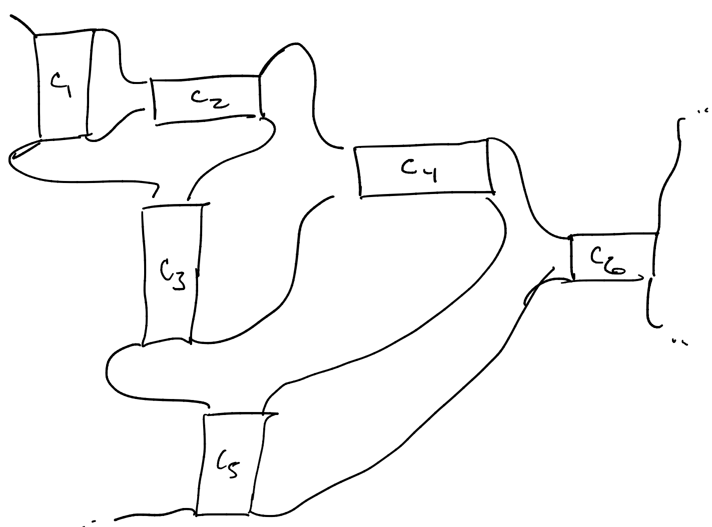
Kauffmann - Lambropoulou (2004)

Def: A rational tangle diagram (RTD) is a tangle diagram obtained from ⌣⌢ (0-tangle) by twisting it and then alternately twisting the bottom endpts then the right endpts some odd # of times in total. We also allow ourselves to start with the ∞ tangle)(and alternate horiz., then vertically an <u>even</u> # of times. So we get one of two templates:

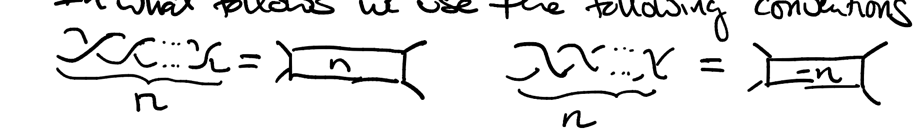
<u>odd case</u>

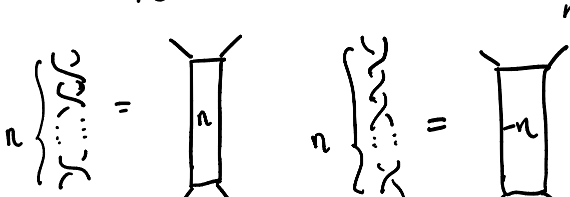


<u>even case</u>

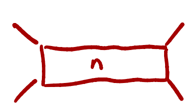


The numbers $c_1, c_2, \dots, c_n$ indicate the number of crossings in each twist box.
In what follows we use the following conventions

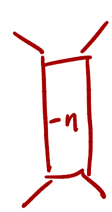




Note:



Rotated 90° becomes



Def The Conway notation for RTDs on the previous page is $[c_1, c_2, \ldots, c_n]$. The Conway number for such an RTD is then the rational #:

$$c_n + \cfrac{1}{c_{n-1} + \cfrac{1}{c_{n-2} + \cdots + \cfrac{1}{c_1}}}.$$

Ex

Conway notation: $[5, -3, -4]$

Conway number: $-4 + \cfrac{1}{-3 + \cfrac{1}{5}} = -4 - \frac{15}{14} = -5\frac{1}{14}$

Fact 1: Every rational number has a continued fraction expansion

Ex $\frac{39}{17} = 2 + \frac{5}{17} = 2 + \frac{1}{17/5} = 2 + \cfrac{1}{3 + \frac{2}{5}}$

$= 2 + \cfrac{1}{3 + \cfrac{1}{5/2}} = 2 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{1}{2}}}$

Thus, for every rational # r there is at least one RTD w/ Conway # r.

<u>Fact 2</u>: Every rational # has a unique continued fraction expansion $[a_1, a_2, \ldots, a_n]$ s.t. n is odd and all a_i are non-zero and have the same sign.

<u>proof</u> Suppose $r \in \mathbb{Q}$ has c.f. expansions $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_m]$ w/ all a_i of same sign and all b_j of the same sign and both n, m odd and no a_i, b_j equal to 0.

WLOG, Assume $n \leq m$. Notice

$$r = a_n + \cfrac{1}{a_{n-1} + \cdots + \cfrac{1}{a_1}} = b_m + \cfrac{1}{b_{m-1} + \cdots + \cfrac{1}{b_1}}$$

If all $a_i > 0$ and $n \geq 2$, $0 < \cfrac{1}{a_{n-1} + \cdots + \cfrac{1}{a_1}} \leq \frac{1}{a_{n-1}} < 1$

Thus, if all $a_i > 0$, then $r \in [a_n, a_n + 1)$.

If all $a_i < 0$, then $r \in (a_n - 1, a_n]$.

Similar results hold for the b_j so all the a_i <u>and</u> b_j have the same sign. Furthermore, $a_n = b_m$.

The result follows by induction. $\square$

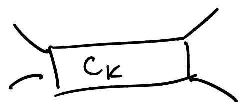
<u>Theorem (Conway)</u>

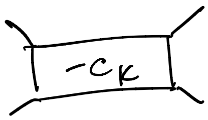
Two RTD represent equivalent tangles iff their Conway numbers are equal.

We give a proof due to Kauffman and Lambropoulou from 2004.

Def If D is a RTD, we let $-D$ denote the same diagram but with all crossings reversed.

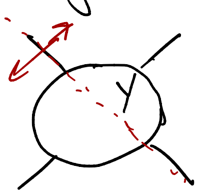
Prop If D has Conway notation $[c_1, \ldots, c_n]$ then $-D$ has Conway notation $[-c_1, \ldots, -c_n]$.

pf Changing the crossings in 

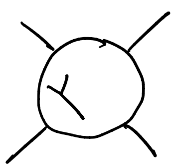
produces 

$\square$.

Def If D is a RTD let $\frac{1}{D}$ denote the diagram obtained by reflecting like so



D



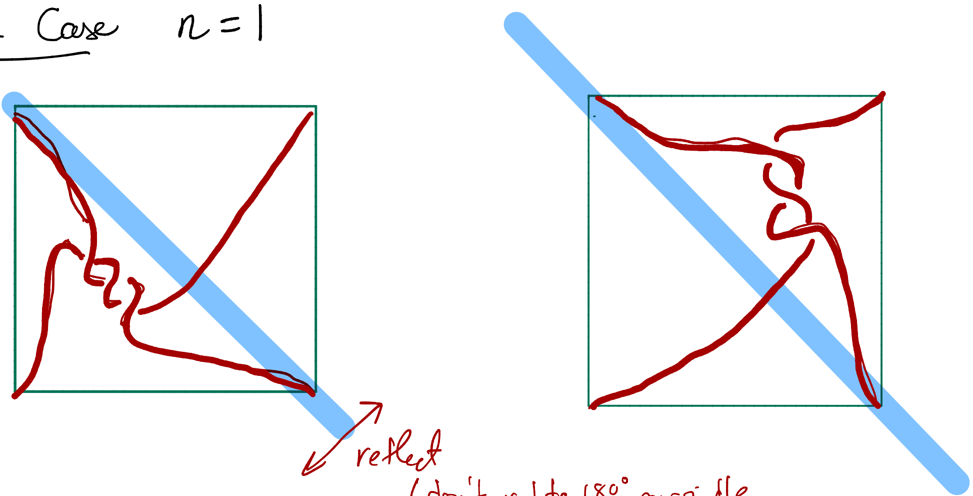
$1/D$

Prop $-\frac{1}{D}$ is the diagram obtained by rotating D $90°$ to the left.

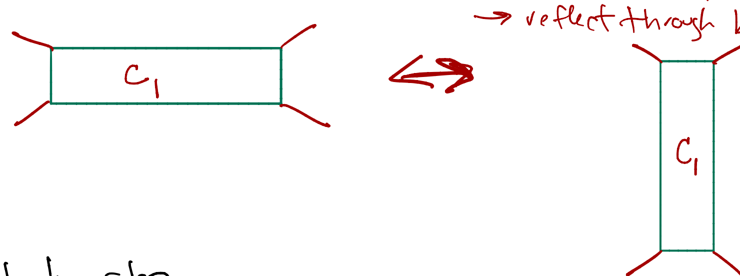
pf On next page.

pf Induct on n.

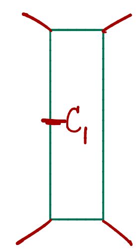
Base Case $n=1$



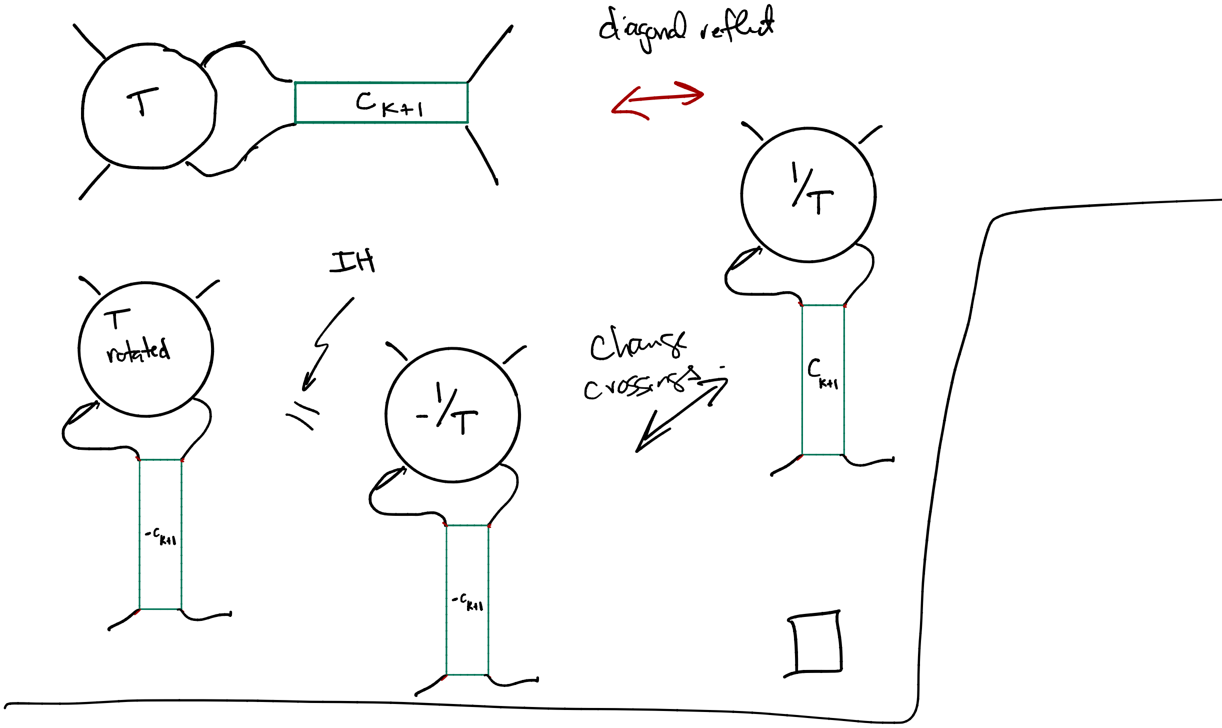
reflect (don't rotate 180° on spindle → reflect through blue mirror)



Changing crossings then gives the rotated version



Inductive step



$\square$

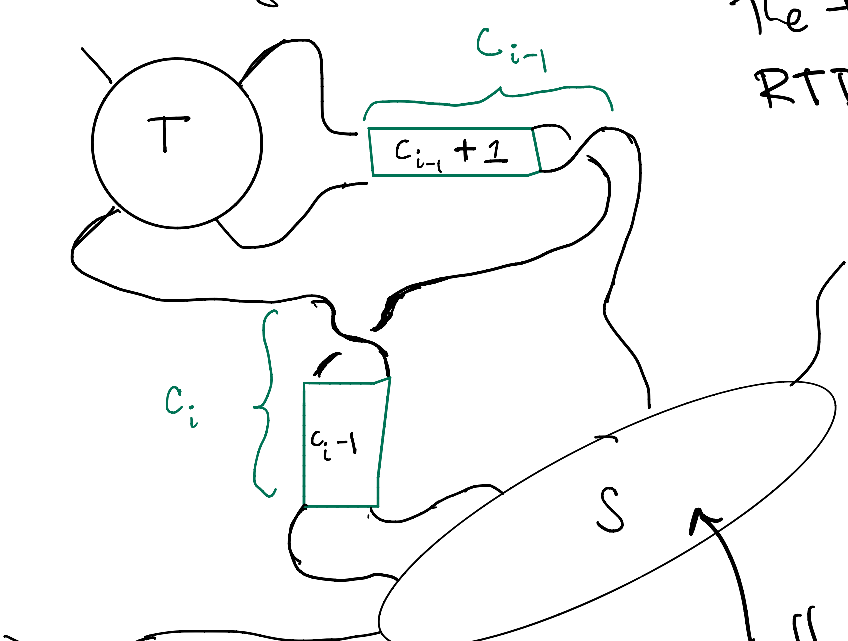
Def For a continued fraction $[c_1, c_2, \ldots, c_n]$ let $I = \max\{i \mid c_i \text{ has a different sign from } c_{i-1}\}$ or 1 if max doesn't exist.

Let $S = |c_1| + \cdots + |c_n|$

Define $c([c_1, \ldots, c_n]) = (I, S)$ and compare lexicographically (e.g. $(5,10) < (6,2)$)

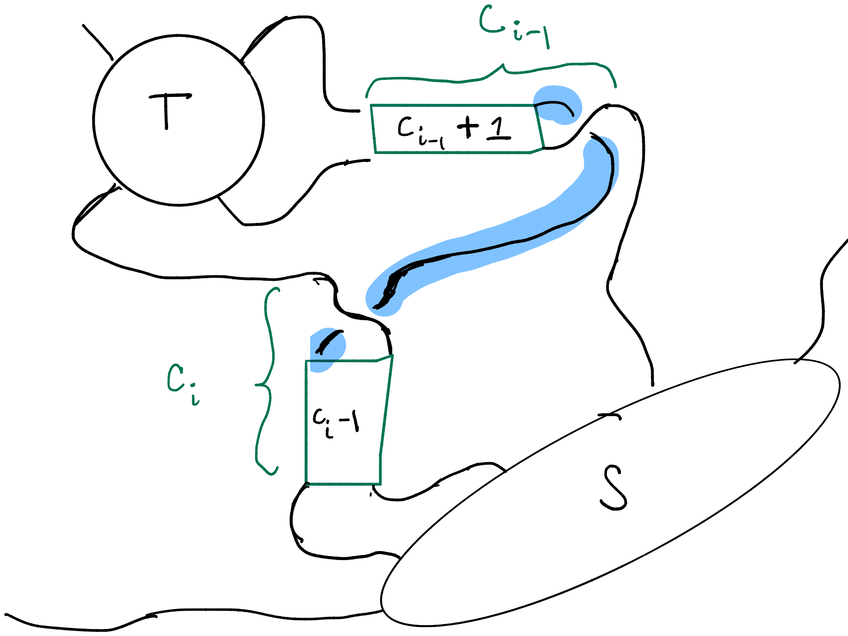
Prop Suppose that D is a RTD s.t. out of all RTD equivalent to D, the complexity $c([c_1, \ldots, c_n])$ is minimized, where $[c_1, \ldots, c_n]$ is the Conway notation for D. Then all c_i have the same sign.

pf Suppose not. Then $I > 1$. For convenience, set $i = I$. If $c_i < 0$, replace D with $-D$. So WLOG, assume $c_i > 0$ and $c_{i-1} < 0$.
For convenience, we'll also assume i is even.
We have the following picture



The tangle T is the RTD w/ Conway notation $[c_1, c_2, \ldots, c_{i-2}]$

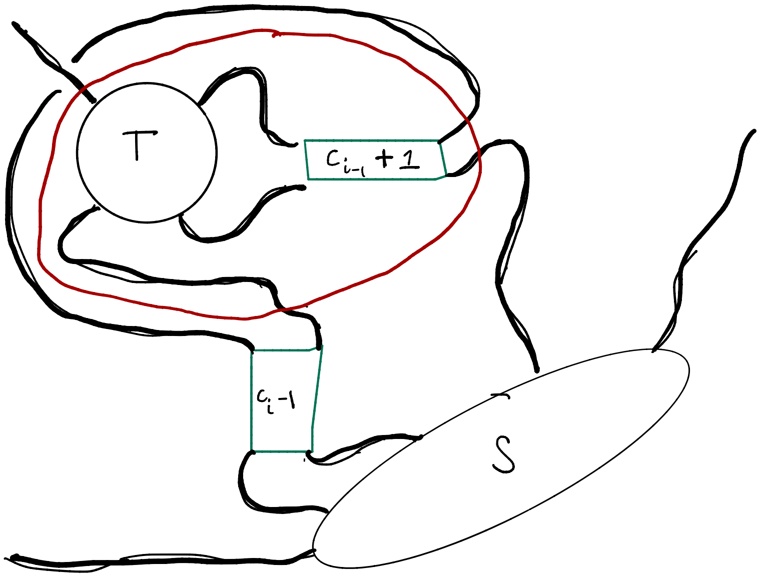
all crossings c_j for $j > i$ in here



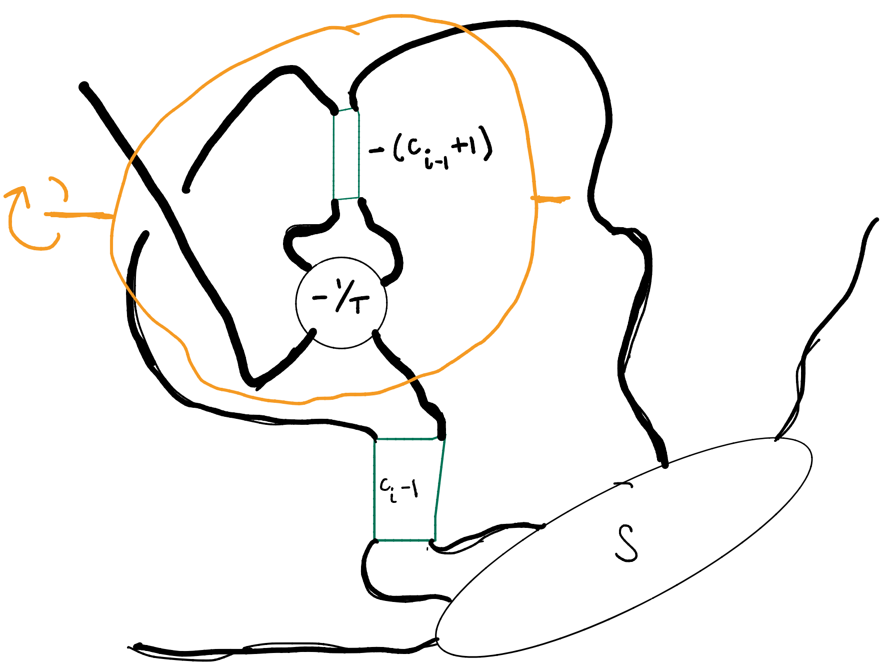
observe the strand highlighted.

Assume both $c_{i-1} < -1$
and $c_i > 1$

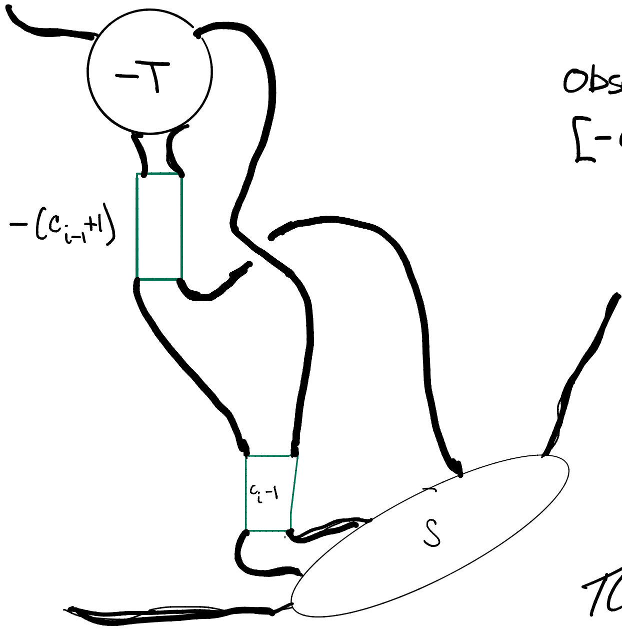
Perform the move
we call a
"loose hair flip"



Now rotate the circled
tangle 90°
counterclockwise



Flip 180° around
axis as
indicated.

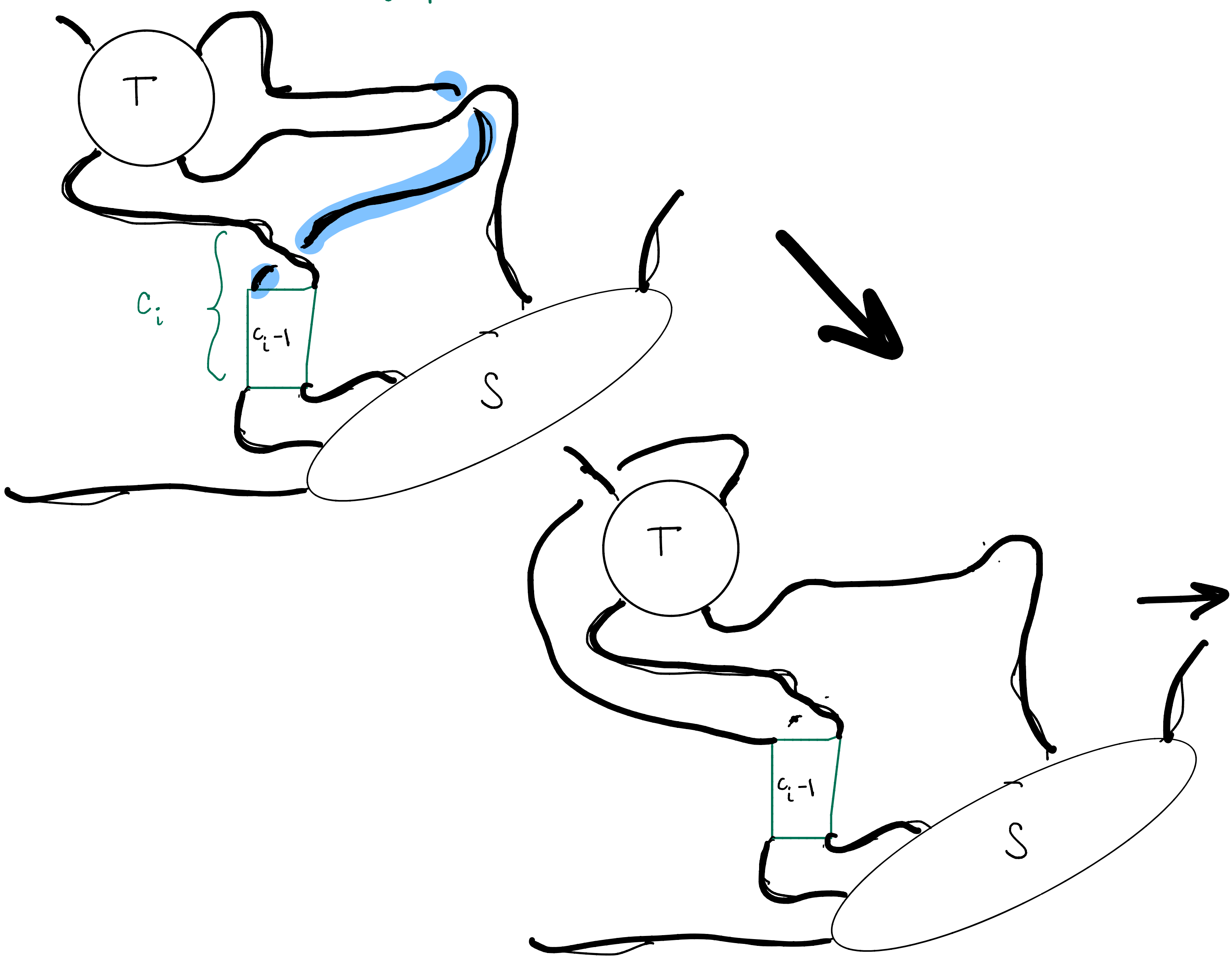


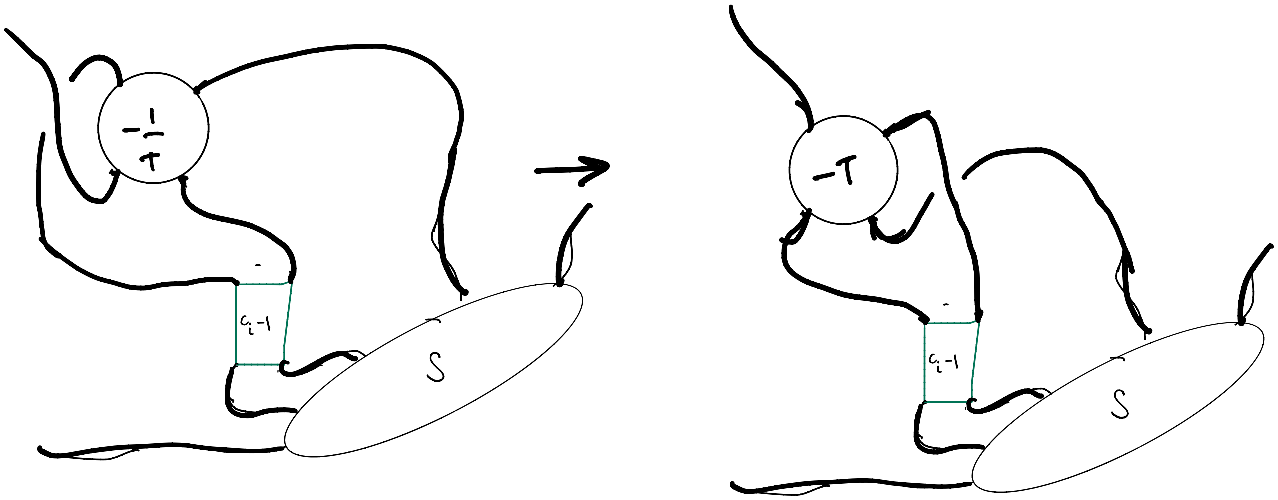
Observe this is a RTD w/ Conway notation
$[-c_1, -c_2, \ldots, -c_{i-2}, -(c_{i-1}+1), 1, c_i-1, c_{i+1}, \ldots, c_n]$

The max k s.t. c_{k-1} and c_k have different signs is still I but S has decreased by 1 since $c_{i-1} < 0$ and $c_i > 0$

This contradicts our choice of D to minimize complexity $\otimes$

Now assume $c_{i-1} = -1$.





This last tangle has Conway notation $[c_1, \ldots, -c_{i-2}+1, c_i - 1, c_{i+1}, \ldots, c_n]$ since i and n are both odd. So I has decreased by 1. This contradicts our choice of RTD

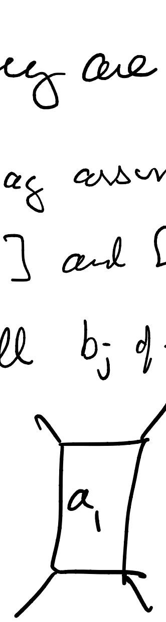
$\square$

Important observation: Notice that in the proof when we convert $[c_1, \ldots, c_n]$ to $[-c_1, \ldots, -c_{i-1}, 1, c_i - 1, c_{i+1}, \ldots, c_n]$ or $[-c_1, \ldots, -c_{i-2}+1, c_i - 1, c_{i+1}, \ldots, c_n]$ the associated Conway number is unchanged

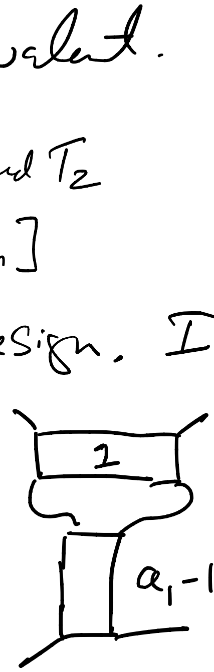
Conways Theorem Part A

If RTD T_1 and RTD T_2 have the same Conway number then they are equivalent.

pf By the previous lemma, we may assume T_1 and T_2 have Conway notations $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_m]$ with all a_i of the same sign and all b_j of the same sign. If n (say) is even and $a_1 \geq 2$, convert

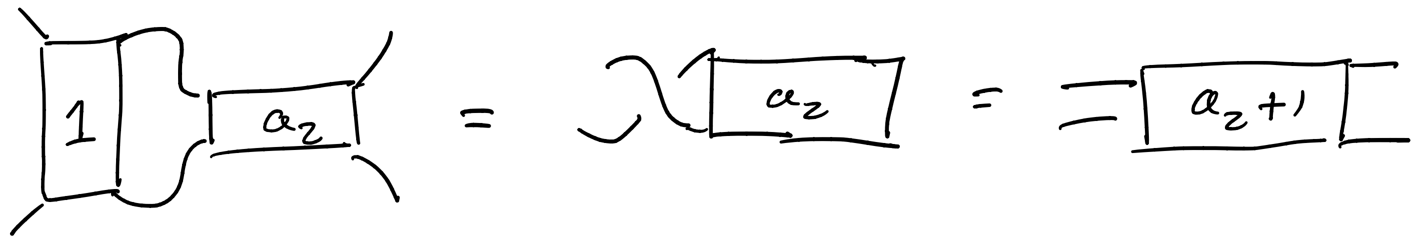


into



so that $[a_1, \ldots, a_n]$ becomes $[1, a_1 - 1, \ldots, a_n]$.

If n is even and $a_1 = 1$ then rewrite as



this converts $[a_1, \dots, a_n]$ into either

$[a_2+1, a_3, \dots, a_n]$. Note that $a_2 > 0$ by hypothesis.

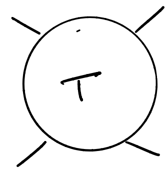
Similar arguments show that we may assume not only that all a_i have the same sign and all b_j have the same sign, but also that n and m are odd.

In which case since the Conway numbers are the same, $n = m$ and $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_n = b_n$. In which case $T_1 = T_2$ as desired. □

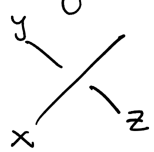
We now prove

Conway's Thm Part B

If T_1 and T_2 are RTDs that are equivalent then their Conway numbers are the same.

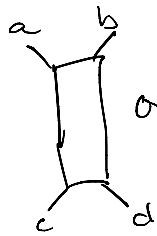
<u>Def</u> Given a 2-strand tangle 

a $\mathbb{Z}$-coloring is an assignment of an integer to each strand s.t. at every crossing if the colors are x, y, z like so

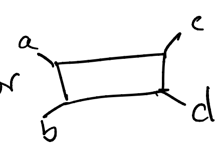


then $2x - y - z = 0$

Observe that for a twist box

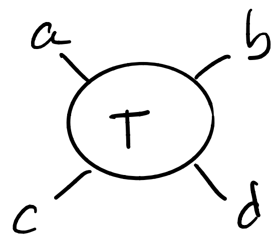


or



if we the entry strands are colored a, b then the coloring can be extended to all strands in the twist box, so c, d exist and are completely determined by a, b. By proceeding one twist box at a time

if T is a rational tangle diagram then given $a, b \in \mathbb{Z}$ $\exists$ a $\mathbb{Z}$-coloring of T w/ top two strands colored a, b. Let c, d be the colors of the bottom two strands:



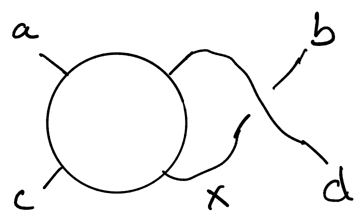
unless $T =$ 

let $M(T)(a,b) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the associated matrix.

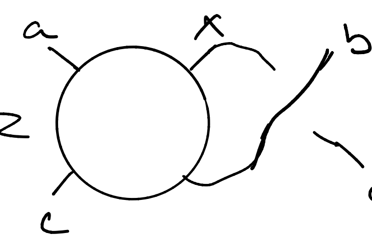
<u>lemma</u> for such a matrix $M(T)(a,b) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $a + d = b + c$

<u>proof</u> It's true for $)($ and $\asymp$. Suppose its true for all RTD with K crossings and that D is a RTD w/ $K+1$ crossings

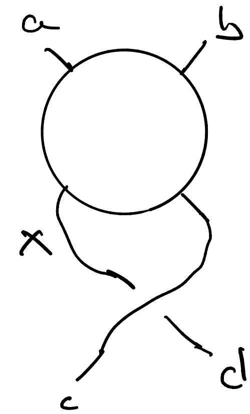
either



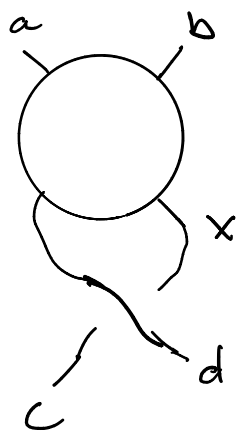
OR



OR



OR



By hypothesis, in the 4 cases we have

$a + x = d + c$, $a + b = x + c$,

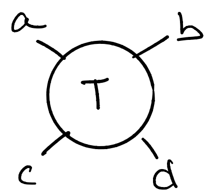
$a + c = b + x$, OR $a + x = b + d$.

Using the crossing eqn. we conclude in each case that $a + d = b + c$. $\square$

Lemma If $M(T)(a,b) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\forall\, n,k \in \mathbb{Z}$ there is a coloring s.t. $\begin{pmatrix} na+k & nb+k \\ nc+k & nd+k \end{pmatrix}$ is a matrix for a valid $\mathbb{Z}$-coloring of T.

pf Scaling a coloring is a coloring & adding two colorings is a coloring.

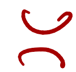
Def The coloring ratio for

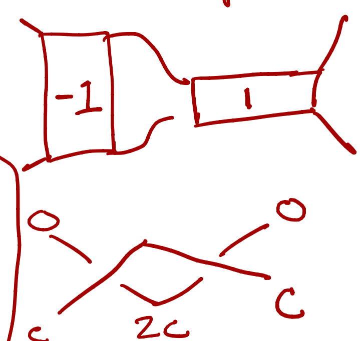


is

$$f = \frac{b-a}{b-d}$$

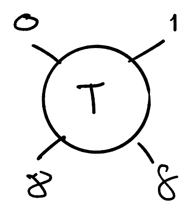
Special Case

Define $f=0$ for  b/c that's what we get for



lemma f depends only on T.

pf Consider the coloring of T



Call it C. Observe that

$$M(T)(a,b) = M(T)\big((b-a)\cdot 0 + a,\ (b-a)\cdot 1 + a\big)$$

In particular, $d = (b-a)\delta + a$. Thus,

$$\frac{b-a}{b-d} = \frac{b-a}{b-((b-a)\delta + a)} = \frac{(b-a)(1-0)}{(b-a)(1-\delta)}$$

$$= \frac{1-0}{1-\delta}.$$

Thus, f does not depend on a,b so we write $f = f(T)$.

Corollary If T_1 is equivalent to T_2 (both RTD) then $f(T_1) = f(T_2)$.

pf a,b,c,d unchanged by Reidemeister moves & f does not depend on the specific coloring. $\square$

Thus, if we show $f(T)$ is equal to the Conway number, then we'll know that any two RTD that are equivalent have the same Conway # (since they have the same value for f

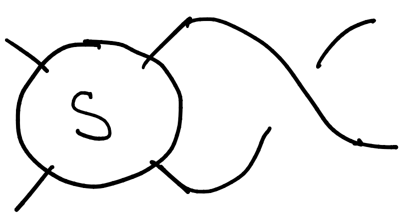
Note for)($f = \infty$ and for $\asymp$ f is defined to be 0.

Prop If T is a RTD then $f(T)$ is the Conway number.

pf We induct on crossing number.

Base Cases)(and $\asymp$ by definition!

Inductive

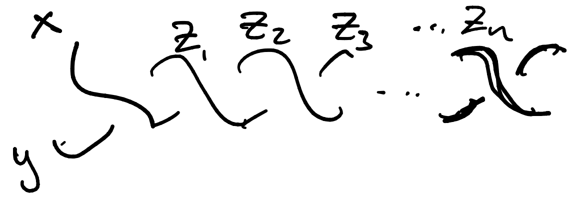


$= T$

Assume S has Conway notation $[s_1, \ldots, s_n]$. Then T has Conway notation

$[s_1, \ldots, s_n + 1]$.

pf Base Cases.



n+1 crossings

Notice $z_1 = 2x - y$

$z_2 = 2z_1 - x = 4x - 2y - x = 3x - 2y$

$z_3 = 2z_2 - z_1 = 4z_1 - z_1 - 2x$

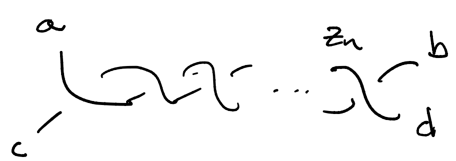
$= 3z_1 - 2x$

$= 4x - 3y$

$\vdots$

$z_n = (n+1)x - ny$

so



coloring has

$(n-1)c - 2nc$

$= -nc - c$

$d = (n+1)a - nc$

$b = 2z_n - z_{n-1} = 2((n+1)a - nc)$

$- (na - (n-1)c)$

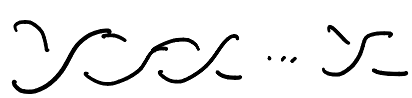
$= na + 2a - (n+1)c$

$= (n+2)a - (n+1)c$

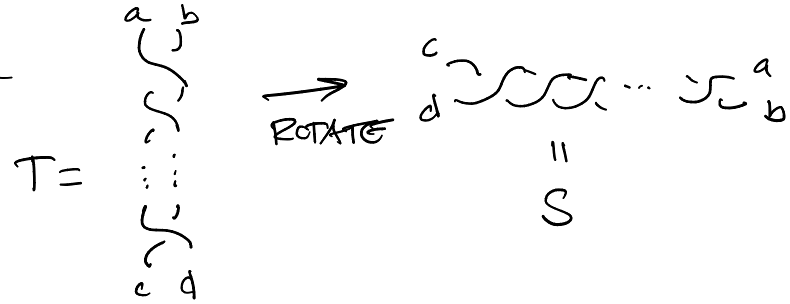
Set $c = 0$, $a = 1 \Rightarrow d = (n+1)$ $b = (n+2)$

so $f = \dfrac{b-a}{b-d} = \dfrac{n+1}{1} = n+1 = \#$ crossing

$=$ Conway #

Similarly it works for 

Now Consider



If there are n crossings we have

$$-n = f(S) = \frac{a-c}{a-b}$$

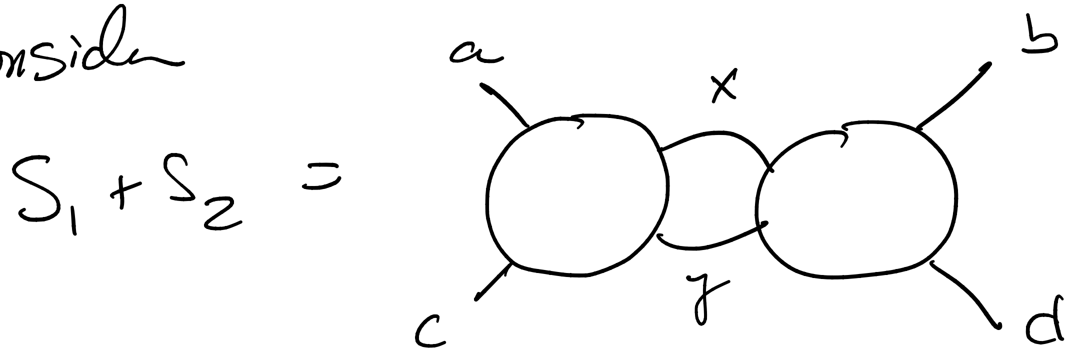
previous case

$$\text{so } -\frac{1}{n} = \frac{b-a}{c-a} = \frac{b-a}{a+d-b-a} = \frac{b-a}{-(b-d)}$$

$$= -f(T).$$

So it works in this case also.

Consider



Assume it is colored as indicated

$$\text{Notice} \quad f(S_1) = \frac{x-a}{x-y}$$

$$f(S_2) = \frac{b-x}{b-d}.$$

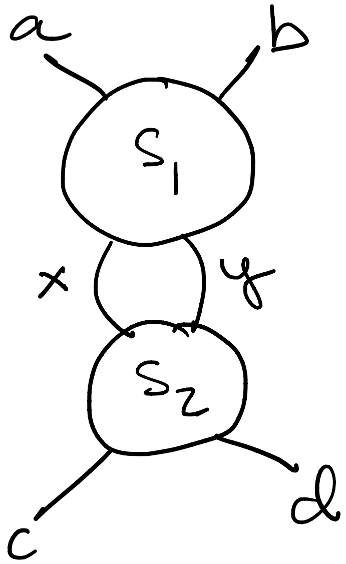
$$f(S_1) + f(S_2) = \frac{b-x}{b-d} + \frac{x-a}{x-y}$$

and
$$a + y = x + c$$
$$x + d = b + y$$

$$\Rightarrow \quad x - y = c - a$$
$$x - y = b - d$$

so $$f(S_1) + f(S_2) = \frac{b-x + x-a}{b-d} = f(S_1 + S_2)$$

Similarly



$$f(S_1) = \frac{b-a}{b-y}$$

$$f(S_2) = \frac{y-x}{y-d}$$

$$a + y = b + x \Rightarrow y - x = b - a$$

so $$\frac{1}{f(S_2)} = \frac{y-d}{y-x} = \frac{y-d}{b-a}$$

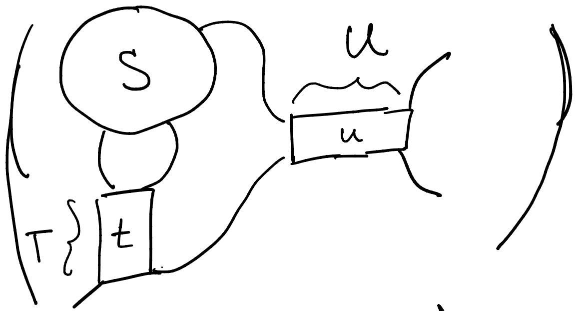
$$\frac{1}{f(S_1)} = \frac{b-y}{b-a} \Rightarrow \frac{1}{f(S_1)} + \frac{1}{f(S_2)} = \frac{b-d}{b-a}$$

$$= \frac{1}{f(S_1 * S_2)}$$

We see that

$$f(S_1 * S_2) = \cfrac{1}{\cfrac{1}{f(S_1)} + \cfrac{1}{f(S_2)}}$$

Consider

f 

$$= f\big((S*T)+U\big)$$

$$= f(S*T) + f(U)$$

$$= \cfrac{1}{\cfrac{1}{f(S)} + \cfrac{1}{f(T)}} + f(U)$$

$$= \cfrac{1}{\cfrac{1}{f(S)} + t} + u \quad (*)$$

t, u twists

If S has Conway notation $[c_1, c_2, \ldots, c_n]$ (c_n odd) then $(S*T)+U$ has Conway notation

$[c_1, c_2, \ldots, c_n, t, u]$

and Conway #

$$U + \cfrac{1}{t + \cfrac{1}{c_n + \cfrac{1}{\ddots + \cfrac{1}{c_1}}}} \quad (**)$$

So by Induction Hyp.

$$c_n + \cfrac{1}{c_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{c_1}}} = f(S)$$

So we see that by $(*)$ and $(**)$

$f\big((S*T)+U\big)$ is the Conway # for $(S*T)+U$.

A similar argument works for RTD w/ even # of terms.

$\square$

By Induction, f is the Conway # for any RTD w/ odd # of terms in Conway notation