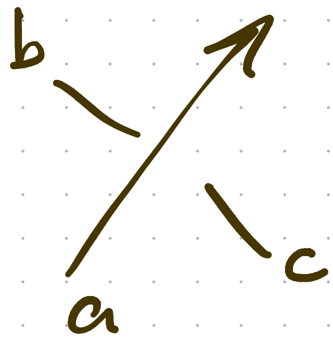
II Combinatorial Knot Group

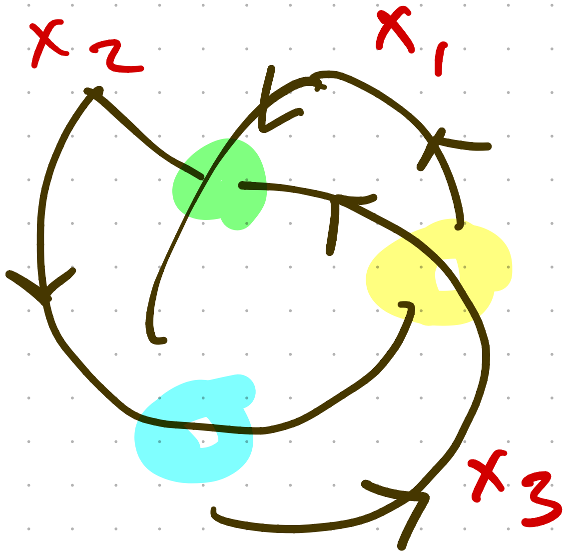
Coloring by a group G

$\{\text{strands}\} \to G$



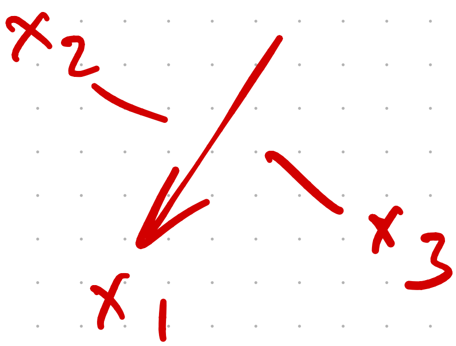
$$c = a^{-1} b a$$
$$\Updownarrow$$
$$\mathbb{1} = \underline{a^{-1} b a c^{-1}}$$

Use diagram D to define a group $G(D)$



Alphabet $= \{ x_1, x_2, x_3 \}$

Relator for each crossing



$$x_2 = x_1^{-1} x_3 x_1$$
$$\Updownarrow$$
$$\mathbb{1} = x_1^{-1} x_3 x_1 x_2^{-1}$$

$$x_3^{-1} x_2 x_3 x_1^{-1}$$

$$x_2^{-1} x_1 x_2 x_3^{-1}$$

$$G(D) = \langle x_1, x_2, x_3 \mid x_1^{-1} x_3 x_1 x_2^{-1},\ x_3^{-1} x_2 x_3 x_1^{-1},\ x_2^{-1} x_1 x_2 x_3^{-1} \rangle$$

$$G(D) = \langle x_1, x_2, x_3 \mid x_1^{-1}x_3x_1x_2^{-1},\ x_3^{-1}x_2x_3x_1^{-1},\ x_2^{-1}x_1x_2x_3^{-1} \rangle$$

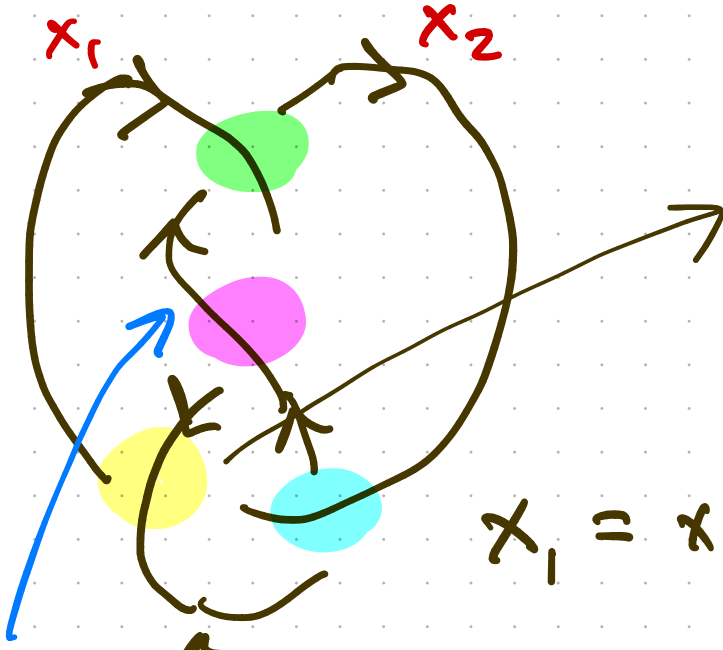
$$= \langle x_1, x_2, x_3 \mid x_2 = x_1^{-1}x_3x_1,\ x_1 = x_3^{-1}x_2x_3,\ x_3 = x_2^{-1}x_1x_2 \rangle$$

$$= \langle x_1, x_2 \mid x_2 = x_1^{-1}x_2^{-1}x_1x_2x_1,\ x_1 = x_2^{-1}x_1^{-1}x_2\cancel{x_2}\cancel{x_2^{-1}}x_1x_2 \rangle$$

$$= \langle x_1, x_2 \mid x_2 = x_1^{-1}x_2^{-1}x_1x_2x_1,\ x_1 = x_2^{-1}x_1^{-1}x_2x_1x_2 \rangle$$

$$= \langle x_1, x_2 \mid x_2x_1x_2 = x_1x_2x_1,\ x_1x_2x_1 = x_2x_1x_2 \rangle$$

$$= \langle x_1, x_2 \mid x_2x_1x_2 = x_1x_2x_1 \rangle$$



$$x_1 = x_2 x_1^{-1} x_2^{-1} x_1 \cancel{x_2^{-1}}$$

$$\cancel{x_2} x_2 x_1^{-1} x_2 x_1$$

$$x_2^{-1}$$

$$\Downarrow$$

$$x_1 = x_2 x_1^{-1} x_2^{-1} x_1 x_2 x_1^{-1} x_2 x_1$$

$$x_2^{-1}$$

$$c = x_1^{-1} x_2 x_1$$

$$c = x_2^{-1} d \, x_2$$

$$\Uparrow$$

$$x_2 c x_2^{-1} = d$$

$$x_2 x_1^{-1} x_2 x_1 x_2^{-1} = d$$

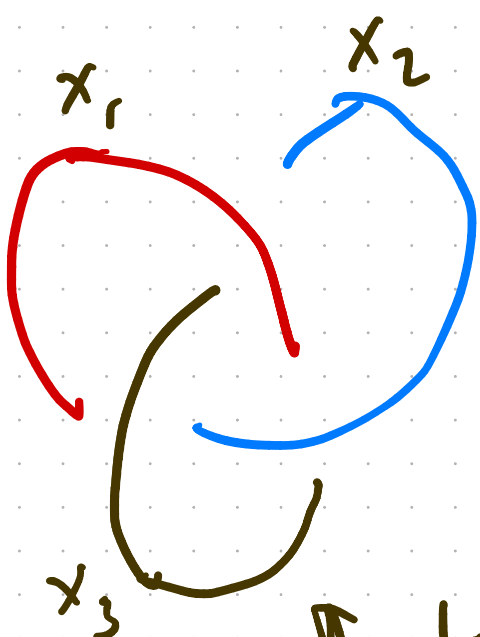
$$G(D) = \langle x_1, x_2 \mid x_1^{-1} x_2 x_1^{-1} x_2^{-1} x_1 x_2 x_1^{-1} x_2 x_1 x_2^{-1} \rangle$$

Know: Every knot has an associated knot group (defined up isomorphism)

Notice if G is some other group that colors a diagram D we have a function

$$G(D) \longrightarrow G$$

strand labelled x_i $\longrightarrow$ color of the strand



↑ coloring by D_3

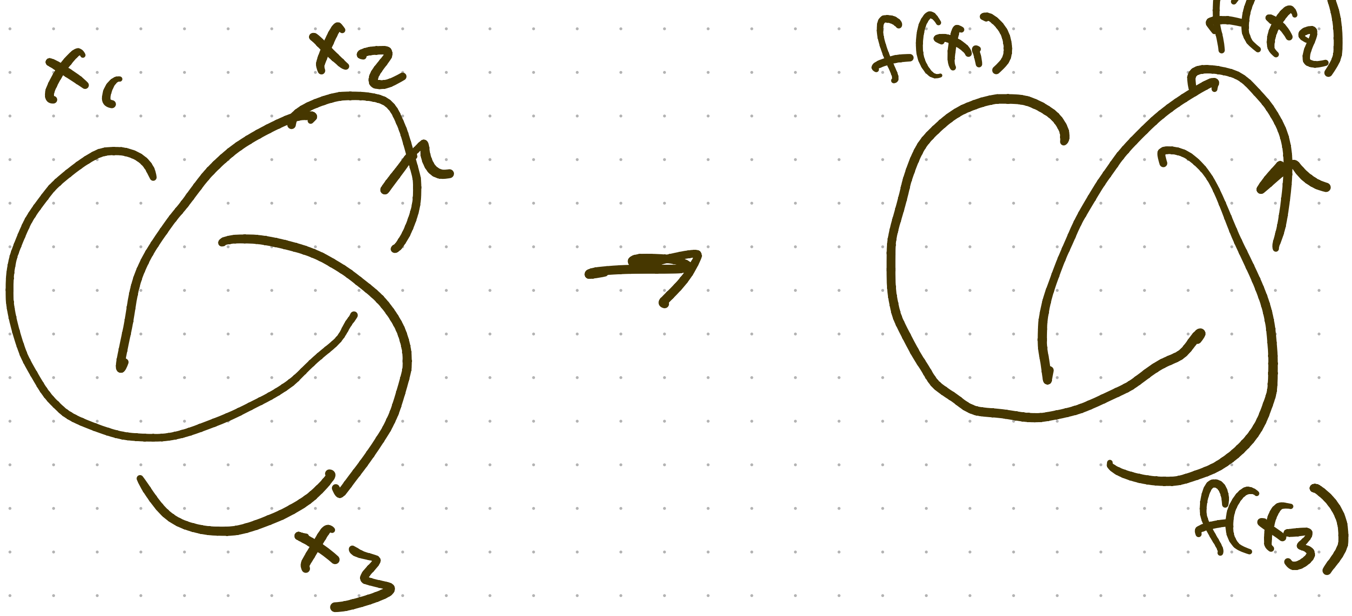
$x_1 \to$ reflection
$x_2 \to$ reflection
$x_3 \to$ reflection

HOMOMORPHISM

Conversely if

if $f : G(D) \to G$

is a homomorphism we get a coloring



$$\begin{array}{c} \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle \\ \big\downarrow f \\ G \end{array}$$

<u>Upshot</u> correspondence between colorings by groups & homomorphisms $G(D) \to G$

Notice that (isomorphism type)

$G(D)$ is a knot invariant

we write $G(K)$ when we don't have a partic. diagram in mind

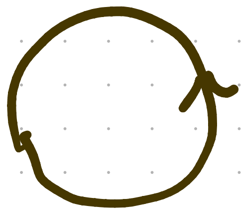
<u>Thm (Gordon - Luecke)</u>

If K_1, K_2 are prime knots with isomorphic knot groups then K_1 is equivalent to K_2 (possibly after mirroring)

<u>Problem</u> knot groups are <u>very</u> hard to analyze

Knotgroups are typically non-abelian (don't commute)

Ex



$$G(D) = \langle x_1 \mid \ \rangle$$

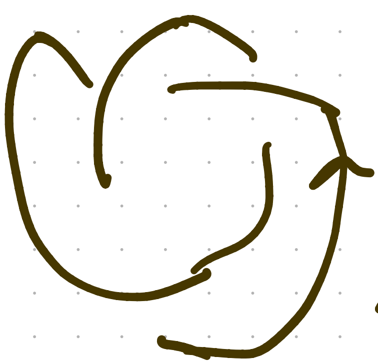
D

$$\cong \mathbb{Z}$$

$$x_1 \to 1$$

$$x_1^2 \to 2 \quad \text{etc.}$$

$$x_1^{-3} \to -3$$



$$\langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

$$\|$$

$$\langle x_1, x_2 \mid x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1} \rangle$$

$\downarrow$ make commute

$$\langle x_1, x_2 \mid x_1^{\cancel{2}} x_1^{-\cancel{1}} \cancel{x_2} x_2^{-\cancel{2}} \rangle \Rightarrow \cong$$

$$\langle x_1, x_2 \mid x_1 = x_2 \rangle = \langle x_1 \mid \ \rangle$$

$$x_i^{-1} x_j x_i = x_k$$

$\downarrow$ make commute

$$x_j = x_k$$

$\Rightarrow$ monochromatic

group becomes $\langle x_1 \mid \ \rangle \cong \mathbb{Z}$

☆ After making a knot group commutative, it becomes $G(O)$

<u>Thm</u> (Papakyriakopoulos)

If $G(K) \cong \mathbb{Z}$ then $K = O$