Notes on the Alexander polynomial and Burau representation.

A <u>Laurent polynomial</u> $p(t)$ is of the form

$$a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0 + a_{-1} t^{-1} + \dots + a_{-m} t^{-m}$$

for some $m, n \in \mathbb{N} \cup \{0\}$. The a_i are the <u>coefficients</u>.

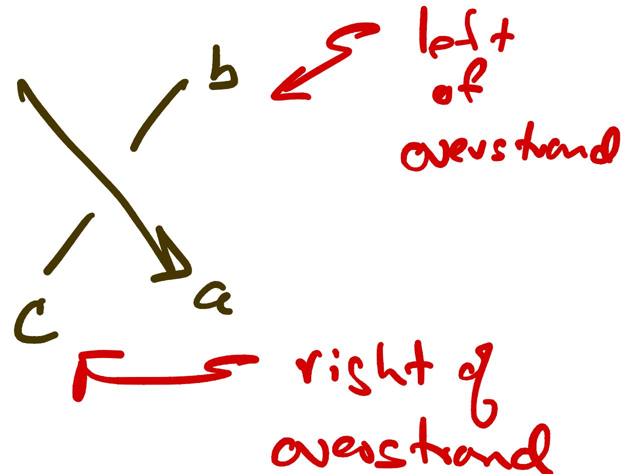
$\mathbb{Z}[t^{\pm 1}]$ is the set of Laurent polynomials w/ integer coefficients. It is a "module" because we can add the elements and scale the elements by integers.

Let $\Lambda = \mathbb{Z}[t^{\pm 1}]$.

$\Lambda^n = \underbrace{\Lambda \times \Lambda \times \dots \times \Lambda}_{n}$ is the set of vectors w/ entries in Λ and n terms.

<u>Ex</u> $(t^2 + t, 1 + \frac{1}{t}) \in \Lambda^2$.

Def An Alexander coloring of a oriented knot, link, braid diagram D is a function $\{\text{strands}\} \to \Lambda$ s.t. at a crossing



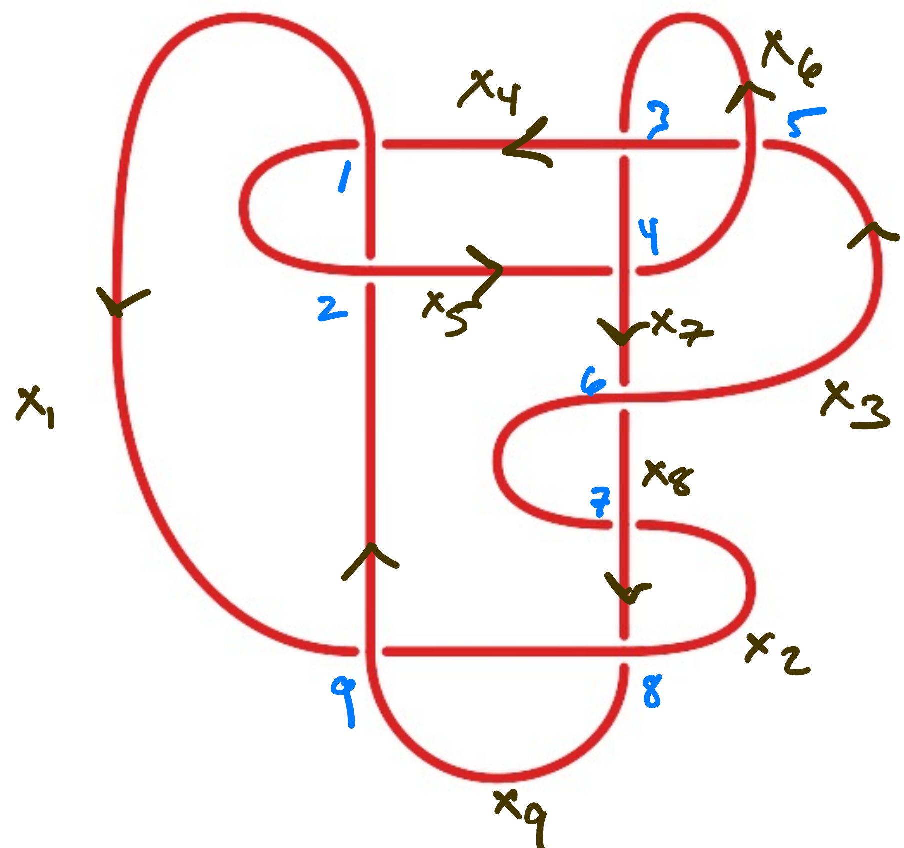
w/ colors a, b, c as indicated we have

$$(1-t)a + tb - c = 0$$

(equivalently, $c = (1-t)a + tb$)

The associated <u>crossing-arc matrix</u> is the matrix corresponding to the linear (for fixed t) system.

EX



9_8 from KnotInfo

1: $(1-t)x_1 + t x_5 - x_4 = 0$

2: $(1-t)x_5 + t x_1 - x_9 = 0$

3: $(1-t)x_4 + t x_7 - x_6 = 0$

4: $(1-t)x_7 + t x_6 - x_5 = 0$

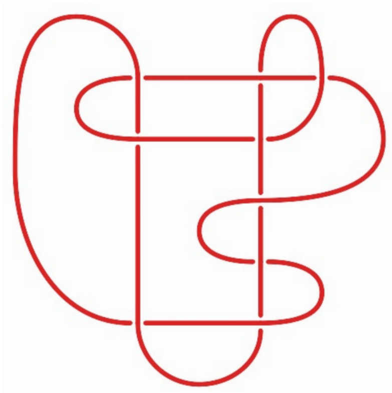
5: $(1-t)x_6 + t x_4 - x_3 = 0$

6: $(1-t)x_3 + t x_7 - x_8 = 0$

7: $(1-t)x_8 + t x_2 - x_3 = 0$

8: $(1-t)x_2 + t x_8 - x_9 = 0$

9: $(1-t)x_9 + t x_1 - x_2 = 0$



1: $(1-t)x_1 + t x_5 - x_4 \overset{9}{=} 0$

2: $(1-t)x_5 + t x_1 - x_9 = 0$

3: $(1-t)x_4 + t x_7 - x_6 = 0$

4: $(1-t)x_7 + t x_6 - x_5 = 0$

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9: $(1-t)x_9 + t x_1 - x_2 = 0$

	1	2	3	4	5	6	7	8	9
1	1-t	0	0	-1	t	0	0	0	0
2	t	0	0	0	1-t	0	0	0	-1
3	0	0	0	1-t	0	-1	t	0	0
4	0	0	0	0	-1	t	1-t	0	0
5	0	0	-1	t	0	1-t	0	0	0
6	0	0	1-t	0	0	0	t	-1	0
7	0	t	-1	0	0	0	0	1-t	0
8	0	1-t	0	0	0	0	0	t	-1
9	t	-1	0	0	0	0	0	0	1-t

Assuming we have a knot diagram we calculate the Alexander polynomial by crossing out any row & any column and finding the determinant

Ex

$$\begin{matrix}
1-t & 0 & 0 & -1 & t & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 1-t & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1-t & 0 & -1 & t & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & t & 1-t & 0 & 0 \\
0 & 0 & -1 & t & 0 & 1-t & 0 & 0 & 0 \\
0 & 0 & 1-t & 0 & 0 & 0 & t & -1 & 0 \\
0 & t & -1 & 0 & 0 & 0 & 0 & 1-t & 0 \\
0 & 1-t & 0 & 0 & 0 & 0 & 0 & t & -1 \\
t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1-t
\end{matrix}$$

$$\det(\quad) = -2t^5 + 8t^4 - 11t^3 + 8t^2 - 2t$$

using Mathematica.

B/c we want this to be invariant under R moves this is only well defined up to mult. by $\pm t^n$ for any $n \in \mathbb{Z}$.

The standard form of the Alexander polynomial is the one w/ positive constant term.

ex $-2t^5 + 8t^4 - 11t^3 + 8t^2 - 2t$

$\downarrow$ multiply by $-t^{-1}$

$2t^4 - 8t^3 + 11t^2 - 8t + 2$

Alexander poly. of knot 9_8 in standard form.

Knot Info lists the Alexander polynomial

as $\underbrace{\text{2-8*t+ 11*t^2-8*t^3+ 2*t^4}}$

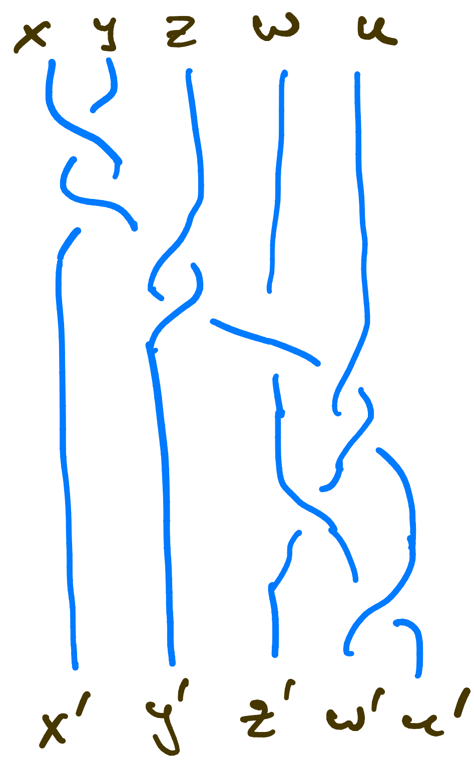
$2 - 8t + 11t^2 - 8t^3 + 2t^4$ So we did it correctly!

plugging in $t = -1$ gives the determinant (up to $\pm$) :

$2 + 8 + 11 + 8 + 2 = 31$

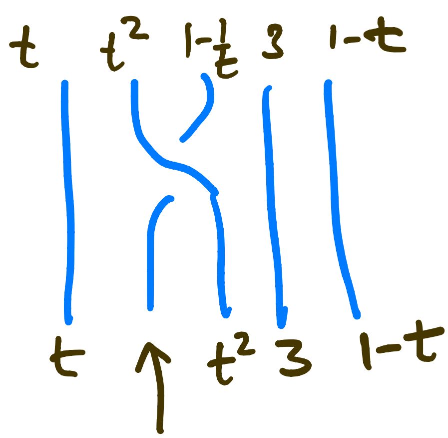
(II) Burau Rep. of Braid groups

Consider a braid on n-strands



Assigning an element of Λ to the top of each strand & orienting the strands downward, we can apply the Alexander coloring rule at each crossing to get elements of Λ assigned to the bottom of each string.

ex



$$(1-t)t^2 + t\left(1-\tfrac{1}{t}\right)$$
$$= -t^3 + t^2 + t - 1$$

So for each n-strand braid β, we get a function

$$f_\beta : \Lambda^n \to \Lambda^n$$

This function is unchanged by Reidemeister moves. It is also <u>linear</u>

$$f_\beta(\vec{v}+\vec{w}) = f_\beta(\vec{v}) + f_\beta(\vec{w})$$

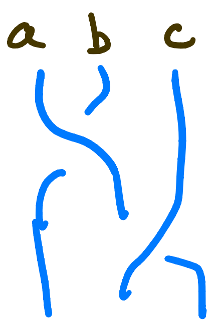
$\forall\ \vec{v},\vec{w} \in \Lambda^n$

and $f_\beta(a\vec{v}) = a f_\beta(\vec{v})$

$\forall\ a \in \mathbb{Z}$ and $\vec{v} \in \Lambda^n$.

So f_β can be represented by a matrix w/ entries in Λ,

<u>Ex</u>



$(1-t)a + tb$ $\quad c \quad$ $(1-t)c + ta$

$$f_\beta\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} (1-t)a + tb \\ c \\ ta + (1-t)c \end{pmatrix}$$

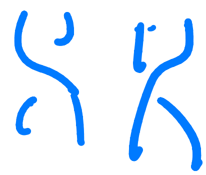
So the matrix is

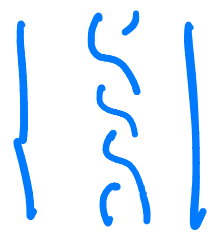
$$\begin{pmatrix} 1-t & t & 0 \\ 0 & 0 & 1 \\ t & 0 & (1-t) \end{pmatrix} = [f_\beta]$$

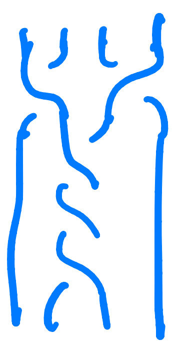
* This has nothing to do w/ a crossing-arc matrix!

Let $B_n = \{ n\text{-strand braids} \}$.

B_n is a group w/ stacking as the operation.

e.g. in B_4: $A =$ 

$B =$ 

$BA =$ 

Note that $\forall\ A, B \in B_n$

braid stacking $\quad f_{BA} = f_B \circ f_A \quad$ function composition

So

$$[f_{BA}] = [f_B][f_A]$$

matrix mult.

Conclusion

$\forall\ n \geq 2$ we have a homomorphism

$$B_n \to GL_n(\Lambda)$$

braids w/ n-strands

$n \times n$ matrices w/ entries in Λ and nonzero determinant

This is called the Burau representation of the braid group.

Sadly it is not injective for $n \geq 5$. (Moody, Long-Paton, Bigelow)

It is injective for $n = 2, 3$

It is UKNOWN if it is injective for $n = 4$.