Fall 2018/MA 331 HW 9: Back to the Fundamentals.

The following must be completed by the end of classes. The reading and problems closely track what we are doing in class.

1. Reading

(1) Read Sections 10.1. and 10.2. In example 10.1, just pay attention to the definitions. Answer the following in an email sent with the subject line: "HW 9 Homotopy Reading"
   (a) Explain the difference between isotopy, ambient isotopy, and homotopy.
   (b) Summarize the main point of Example 3 in Section 10.2.

(2) Read Section 10.3 and 10.4. Answer the following in an email sent with the subject line: "HW 9 Fundamental Group Reading"
   (a) Briefly explain why to form a group out of paths we need to consider paths up to the equivalence relation of "basepoint-preserving homotopy".
   (b) Explain how to show that the maps on π_1 induced by continuous functions play nicely with function composition.

(3) Read Sections 14.2 and 14.3, you do not need to work through the details of the proofs. Answer the following in an email sent with the subject line: "HW 9 Classification of Surfaces"
   (a) What is a surface?
   (b) What is an identifying polygon for a surface?
   (c) What is the connected sum of two surfaces?
   (d) List the steps to show that every surface has a "standard" identifying polygon.
   (e) What is the role of the fundamental group in the classification of surfaces?

2. Proofs and Problems

(1) Let X be a path-connected topological space with basepoint $x_0 \in X$. Suppose that α, β, and γ are paths in X each starting and ending at x_0. Define

$$\alpha \cdot \beta \cdot \gamma(s) = \begin{cases} \alpha(3s) & 0 \le s \le 1/3 \\ \beta(3s-1) & 1/3 \le s \le 2/3 \\ \gamma(3s-2) & 2/3 \le s \le 1 \end{cases}$$

for all $s \in [0,1]$. Prove that $\alpha \cdot \beta \cdot \gamma$ is basepoint-preserving homotopic to $\alpha \cdot (\beta \cdot \gamma)$.

(2) Suppose that X and Y are path-connected topological spaces having basepoints x_0 and y_0 respectively. Also suppose that $f : (X, x_0) \to (Y, y_0)$ is continuous. (This means that $f : X \to Y$ is a continuous function and $f(x_0) = y_0$.) Define

$$f_*([\alpha]) = [f \circ \alpha]$$

for all $[\alpha] \in \pi_1(X, x_0)$.

   (a) Prove that f_* is well defined. That is, if $\alpha \approx \alpha'$ then $f \circ \alpha \approx f \circ \alpha'$.
   (b) Prove that if Z is a path-connected topological space with basepoint z_0 and if $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ are continuous, then

$$g_* \circ f_* = (g \circ f)_*.$$

   Conclude that if X and Y are homeomorphic via a homeomorphism taking x_0 to y_0, then $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic.

(3) (The Straightening Technique) Suppose that X is a topological space and that $A \subset X$ is a subspace such that there is a homeomorphism $\phi\colon A \to C$ where $C \subset \mathbb{R}^n$ is a closed convex set (such as a line segment). Let $\alpha\colon [0,1] \to X$ be a path whose range intersects A but with the property that $\alpha(0), \alpha(1) \notin A$. Suppose also that there exist $0 < s_0 < s_1 < s_1 < 1$ such that $\alpha(s) \in A$ for all $s \in [s_0, s_1]$. Prove that there is a homotopy F from the path α to a path α' such that:

- For all $s \notin [s_0, s_1]$, we have $\alpha'(s) = \alpha(s)$.
- The range of $\phi \circ \alpha'\big|_{[s_0,s_1]}\colon [s_0, s_1] \to C$ is either a straight line parameterized monotonically (i.e. no backtracking) or is constant.
- For all $s \notin [s_0, s_1]$ and all $t \in [0,1]$, $F(s,t) = \alpha(s)$.

(4) Do Problem 4 from Section 10.3.

(5) Do Problem 4 from Section 10.4.

(6) Use the fundamental group to prove that there is no function $r\colon D^2 \to S^1$ (where $D^2 \subset \mathbb{R}^2$ is the unit disc) such that $r(p) = p$ for every $p^{\in} S^1$.

(7) Do Problems 2,3, and 7 from Section 14.3.