Fall 2018/MA 331 HW 8: Was the Mayflower compact?

Before beginning this homework assignment, please review the guidelines for submitting homework. In particular, **If you consult a classmate or online source,** you must give credit for the help you received. Failure to do so may result in a report of academic dishonesty. You are, however, strongly encouraged to work with classmates – just be sure to give them credit for any ideas or help they provide!

Also, please write down the total amount of time spent working on the assignment at the top of what you turn in. If you are spending significantly more than 8 hours per week on homework assignments, you should talk with me to devise a new strategy.

The weekly homework assignments are broken out by day. It is crucial that you meet the deadlines for the reading assignments. When you do the reading, I encourage you to try to prove the theorems/propositions/etc. for yourself before reading the proofs in the book. As you read, sketch additional pictures, make marginal notes. In other words, be an active reader!

For the problems, I strongly encourage you to work with classmates, but be sure you are an active contributor to the discussion. Do not spend time looking for additional online sources. It is easy to waste a lot of time which could be used thinking. There are also a lot of proofs out there which are incorrect or which require a different background from what you have or assume that the course is structured differently.

There is no additional reading for this week. The following problems are mostly all fairly challenging and you should start early. They are intended to give you the opportunity to put together many of the tools we've discussed so far. If you are feeling overwhelmed by the assignment, come get help earlier rather than later.

Note: Corrections are in red.

(1) Let X be a topological space. Prove that X has at least k connected components if and only if there exists a surjective continuous function $X \to \{0,\dots,k-1\}$.

(2) (1st steps in Algebraic Topology after Kosniowski) Recall that $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ is the group with two elements. We can multiply and add elements of $\mathbb{Z}/2\mathbb{Z}$ using 0 as the additive identity, and 1 as the multiplicative identity and $1+1=0$.

For a topological space X, let $C^0(X) = \{f : X \to \mathbb{Z}/2\mathbb{Z}\}$ be the set of all continuous functions $X \to \mathbb{Z}/2\mathbb{Z}$.

(a) Prove that $C^0(X)$ is a vector space. That is:

(i) If $f, g \in C^0(X)$, then $f + g \in C^0(X)$
(ii) If $k \in \{0,1\}$ and $f \in C^0(X)$, then $kf \in C^0(X)$.
(iii) Notice that the function which is constantly 0 is the additive identity for $C^0(X)$.

(b) Let B be a basis for $C^0(X)$. This means that $B \subset C^0(X)$, and given $f \in C^0(X)$ there exist functions $b_1, b_2, \dots, b_n \in B$ such that

$$f = b_1 + \cdots + b_n \pmod 2$$

Furthermore, two finite sums of basis functions are equal if and only if exactly the same functions are in each sum. Assume that X has finitely many connected components and prove that B has finitely many elements and that this number is equal to the number of connected components of X.

(BONUS: Do this without the hypothesis that X has finitely many connected components.)

(EASIER VERSION (for less points): Prove that X is connected if and only if B has exactly one elements.)

(3) (Based on Kosniowski) These are the pancake and the antipodal point problems. See the figure below for clarification. They are motivated by the question: given a misshapen pancake and four siblings, is it possible to divide the pancake into four pieces of equal area using only two cuts which are perpendicular to each other?

   (a) Recall that S^n is the set of all unit vectors in $\mathbb{R}^{n+1}$. Using the fact that S^n is connected for $n \geq 1$, show that if $f\colon S^n \to \mathbb{R}$ is continuous, then there exists $x \in S^n$ such that $f(x) = f(-x)$. (Hint: Consider $g\colon S^n \to \mathbb{R}$ defined by $g(x) = f(x) - f(-x)$. If $x_0 \in S^2$ has $g(x_0) > 0$, show that $g(-x_0) < 0$ and use connectedness.)

   (b) Let $A \subset \mathbb{R}^2$ be a a bounded region having well-defined area and smooth boundary. Prove that there exist perpendicular lines L and M in $\mathbb{R}^2$ dividng A (no matter how misshapen) into four parts, each of exactly the same area. Hint:

      (i) Let S be a circle enclosing A. For $x \in S$, let D_x be the diameter of S having an endpoint at x. Prove (using IVT) that for every $x \in S^1$, there exists a line L_x perpendicular to D_x cutting A into two pieces of exactly equal area. Also show that either there is a unique such line or there is a closed interval's worth (on D_x) of such lines. In the latter case, take L_x to pass though the midpoint of the interval.

      (ii) Fix $x \in S^1$ and let y be a point a quarter of the way around the circle from x. Let M_x be the line perpendicular to D_y cutting A into two pieces of exactly equal area. (As before, if there is ambiguity for M_x, then there is an interval's worth and we choose the midpoint of the interval for M_x to pass through.) So L_x and M_x are perpendicular lines each dividing A into two regions of equal area. Number the regions counter-clockwise by $R_1(x), R_2(x), R_3(x), R_4(x)$. Let $g_i(x)$ be the area of $R_i(x)$. Explain why each g_i is continuous. (Hint: move x slightly and argue that the area changes only slightly.)

      (iii) Prove that $g_1 = g_3$ and $g_2 = g_4$.

      (iv) Let $f = g_1 - g_2 = g_3 - g_4$. Let $y \in S^1$ be the point obtained by rotating x $\pi/2$ radians counter-clockwise. Prove that $f(y) = -f(x)$.

      (v) Prove that there exists $x \in S^1$ with $f(x) = 0$.

(4) (Based on Burago, Burago, Ivanov.) Let (X, d) be a metric space and suppose $A \subset X$. For $r > 0$, $N_r(A) = \{x \in X : d(x, A) < r\}$. Equivalently, $N_r(A)$ is the union of all open balls of radius r centered at points of A. For compact subsets $A, B \subset X$, let

$$d_H(A, B) = \inf\{r > 0 : B \subset N_r(A) \text{ and } A \subset N_r(B)\}.$$

   (a) Prove that $d_H(A, B)$ equals the maximum of $\sup\{d(a, B) : a \in A\}$ and $\sup\{d(A, b) : b \in B\}$.

   (b) Prove that $d_H(A, B) \leq r$ if and only if $d(a, B) \leq r$ for all $a \in A$ and $d(A, b) \leq r$ for all $b \in B$.

   (c) Prove that d is a metric on the set $K(X)$ of all compact subsets of X. (This is called the Hausdorff metric, but don't look it up online in order to complete this problem.)

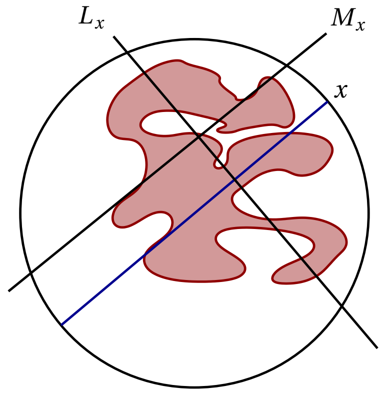

FIGURE 1. The Pancake Problem

(d) Suppose that if X is compact and $A_i \subset X$ is compact for all $i \in \mathbb{N}$. Suppose that for all $i \in \mathbb{N}$, $A_{i+1} \subset A_i$. Prove that the sequence (A_i) in $K(X)$ converges (in the Hausdorff metric) to $\bigcap\limits_{i \in \mathbb{N}} A_i$.

(e) (BONUS) Suppose that $X \subset \mathbb{R}^n$ is compact and that for all $i \in \mathbb{N}$ we have compact $A_i \subset X$. Suppose also that (A_i) converges to $A \in K(X)$ (using the Hausdorff metric). Prove that if each A_i is convex, so is A_i. (That is, the set of all compact, convex sets in X is closed in $K(X)$.)

(5) Suppose that X and Y are topological spaces. Let $C(X,Y)$ be the set of all continuous functions $f\colon X \to Y$. For a compact set $K \subset X$ and an open set $U \subset Y$, let $B(K,U)$ be the set of all elements $f \in C(X,Y)$ such that $f(K) \subset U$. Let $\mathscr{T}$ be the topology (called the **compact-open topology**) on $C(X,Y)$ generated by collection of all $B(K,U)$. (That is, the $B(K,U)$ form a sub-basis for the topology.) Give $C(X,Y)$ this topology. Prove the following:

(a) If X has a single point, then $C(X,Y)$ is homeomorphic to Y.

(b) If Y is Hausdorff, then so is $C(X,Y)$.

(c) Suppose Y is a metric space with metric d and that (f_k) is a sequence in $C(X,Y)$ converging (according to the topology $\mathscr{T}$) to $f \in C(X,Y)$. Suppose also that $K \subset X$ is compact. Prove that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $a \in K$ and all $n \geq N$, $d(f_n(a), f(a)) < \epsilon$. (That is (f_k) converges to f uniformly on K.)

(d) Suppose that both X and Y are metric spaces and that d_Y is the metric on Y and that X is compact. Let $d^*(f,g) = \sup\limits_{x \in X} d_Y(f(x), g(x))$. Show that d^* is a metric on $C(X,Y)$ and that the resulting topology is the compact-open topology.

(6) Let X be a non-compact topological space and let ∞ be some point not in X. Let $\widehat{X} = X \cup \{\infty\}$. Define a subset $U \subset \widehat{X}$ to be **open** if either of the following holds: $U \subset X$ is open in X or there exists a closed, compact set $K \subset X$ such that $U = (X \setminus K) \cup \{\infty\}$. Prove:

(a) this defines a topology on $\widehat{X}$ (called the **one-point compactification** of X) Hint: use the definition of "topology" involving closed sets, rather than open sets.

(b) $\widehat{X}$ with this topology is compact,

(c) For every open set $U \subset \widehat{X}$, there exists $x \in U \cap X$. (That is, X is **dense** in $\widehat{X}$)

(d) Prove that if X is homeomorphic to Y then $\widehat{X}$ is homeomorphic to $\widehat{Y}$.

(7) (Freudenthal Endpoint Compactification) Let $X \subset \mathbb{R}^n$ be a path connected, closed and unbounded subset. Give X the euclidean metric d. Recall that the ball

$$\overline{B}(x_0, R) = \{x \in X : d(x_0, x) \leq R\}$$

is compact.

A **proper ray** in X is a continuous map $r\colon [0,\infty) \to X$ such that for every $R > 0$, there exists $N \in [0,\infty)$ such that for all $t \geq N$, $r(t) \notin \overline{B}(x_0, R)$. Let $\mathscr{R}$ denote the set of all proper rays r such that $r(0) = x_0$. See the figure below.

For $R > 0$ and proper rays $r_1, r_2 \in \mathscr{R}$, we define $r_1 \sim_R r_2$ if and only if there exists $N_1, N_2 \in [0,\infty)$ such that for all $s \geq N_1$ and $t \geq N_2$, there is a path in $X \setminus \overline{B}(x_0, R)$ from $r_1(s)$ to $r_2(t)$.

(a) Prove that for each $R > 0$, $\sim_R$ is an equivalence relation on $\mathscr{R}$.

(b) For $r_1, r_2 \in \mathscr{R}$, define $r_1 \sim r_2$ if and only if $r_1 \sim_R r_2$ for all $R > 0$. Prove that $\sim$ is an equivalence relation on $\mathscr{R}$. Let $\overline{\mathscr{R}}$ denote the set of equivalence classes. An element of $\overline{\mathscr{R}}$ is called an **end** of X.

(c) Let $\widehat{X} = X \cup \overline{\mathscr{R}}$. Declare any open set in X to be **open** in $\widehat{X}$. Also, for each $R > 0$, consider an R-equivalence class $[r]_R$. Let U_r be the component of $X \setminus \overline{B}(x_0, R)$ containing all but a bounded amount of each ray in $[r]_R$. Consider the set $U_r \cup [r]_R$ to be **open** as well.[1] Show that these open sets are a basis for a topology on $\widehat{X}$.

(d) Prove that (with this topology) $\widehat{X}$ is compact.

(e) For $X = \mathbb{R}$, show that $\widehat{X}$ is homeomorphic to $[0,1]$ but for $X = \mathbb{R}^2$, $\widehat{X}$ is (homeomorphic to) the one-point compactification of X.

(f) (BONUS) Let X be an infinite, rooted binary tree without any leaves. Give X the metric where each edge has length one, and the distance between two vertices is the minimum number of edges needed to travel from one vertex to the other along edges of the graph. Show that as a subspace of $\widehat{X}$, the set $\overline{\mathscr{R}}$ is homeomorphic to the Cantor Set.

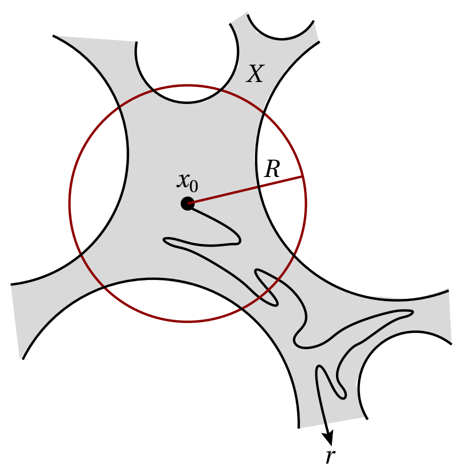

FIGURE 2. A proper ray. At time N the ray leaves the compact ball of radius R forever.

[1] Really we should write $U_r \cup \{[s] : s \in [r]_R\}$ where we restrict to all rays r such that U_r contains a bounded amount of r.