Fall 2018/MA 331 HW 7: Feeling Connected

Before beginning this homework assignment, please review the guidelines for submitting homework. In particular, **If you consult a classmate or online source,** you must give credit for the help you received. Failure to do so may result in a report of academic dishonesty. You are, however, strongly encouraged to work with classmates – just be sure to give them credit for any ideas or help they provide!

Also, please write down the total amount of time spent working on the assignment at the top of what you turn in. If you are spending significantly more than 8 hours per week on homework assignments, you should talk with me to devise a new strategy.

The weekly homework assignments are broken out by day. It is crucial that you meet the deadlines for the reading assignments. When you do the reading, I encourage you to try to prove the theorems/propositions/etc. for yourself before reading the proofs in the book. As you read, sketch additional pictures, make marginal notes. In other words, be an active reader!

For the problems, I strongly encourage you to work with classmates, but be sure you are an active contributor to the discussion. Do not spend time looking for additional online sources. It is easy to waste a lot of time which could be used thinking. There are also a lot of proofs out there which are incorrect or which require a different background from what you have or assume that the course is structured differently.

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1. For Wednesday

1.1. **Reading Assignment.** Read Section 6.5. You do not need to read this section super-carefully. Just get the idea of how the definitions work. You do not need to memorize the definitions in this section.

Answer the following questions in an email sent to me by Tuesday night at 7 PM with the subject line "HW 7 Wednesday Reading Assignment." Memorize the definition of

(1) What does it mean for a space to be locally connected at a point? locally path connected at a point?
(2) Example 4 shows that a non-locally path connected space may be the image under a continuous map of a locally path connected space. (The same is not true if we drop the word "locally") Give an explanation (possibly informal) of why the image of a locally path connected space under a *homeomorphism* will be locally path connected.

1.2. **Standard Proofs and Problems (90%).**

(1) Give a complete proof that the topologists' sine curve is not path connected. If you use any results from 6.5, be sure you understand the proofs.

(2) From Section 6.3: Problems 2, 5.

(3) Recall the comb metric space: (X,d) where $X = \{(x,y) : y \geq 0\}$ and

$$d\big((x,y),(a,b)\big) = \begin{cases} |y-b| & \text{if } x = a \\ |x-a|+|y|+|b| & \text{if } x \neq a \end{cases}$$

Use the idea of connectedness to prove that X is not homeomorphic to $Y = \{(x, y) : y \geq 0\}$ with the euclidean topology.

(4) This problem develops some properties of paths which will be useful down the road. Let X be any topological space and let $I = [0, 1]$ with the standard topology. For a paths $\alpha\colon I \to X$ and $\beta\colon I \to X$, such that $\alpha(1) = \beta(0)$, define the **concatenation** of β and α to be:

$$\beta \cdot \alpha(t) = \begin{cases} \alpha(2t) & \text{if } t \in [0, 1/2] \\ \beta(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

Define the **reverse** of α to be:

$$\overline{\alpha}(t) = \alpha(1-t)$$

(a) Prove that $\beta \cdot \alpha$ is continuous (and hence is a path in X) (i.e. fill in the details of Proposition 2 from 6.3)
(b) Prove that $\overline{\alpha}$ is continuous (and hence is a path in X).
(c) Define $\sim$ on X by declaring $x \sim y$ if and only if there exists a path from x to y. Prove that $\sim$ is an equivalence relation.
(d) The text (p 139) defines a **path component** of X to be a maximal path-connected subset of X. Prove that the path components of X are exactly the equivalence classes of X using the equivalence relation from the previous part.

1.3. **Advanced Proofs and Problems (10%).** In class we proved the polygonal Jordan Curve Theorem. In this problem you will sketch a proof of the stronger theorem: The polygonal Schonflies Theorem. As before, a **polygon** P in $\mathbb{R}^2$ is a subset of $\mathbb{R}^2$ homeomorphic to a circle and which is the union of finitely many line segments intersecting only at their endpoints.

Theorem. *Suppose that $P \subset \mathbb{R}^2$ is a polygon. Then $\mathbb{R}^2 \setminus P$ is disconnected, one component is bounded and is homeomorphic to an open disc whose closure is homeomorphic to a disc. If we use stereographic projection h to project P onto S^2, then $S^2 \setminus h(P)$ is the disjoint union of two subspaces each with closure homeomorphic to a disc.*

We don't actually need the Jordan Curve Theorem to prove the Schoenflies Theorem, but it will simplify matters.

(1) Prove that if $D \subset \mathbb{R}^2$ is a closed subset with boundary a triangle, then D is homeomorphic to a closed disc. (Hint: we discussed this some time ago – think about projecting along rays.)
(2) We now proceed by induction. Suppose that P is a polygon with $n \geq 4$ sides. Prove that there is a line segment α lying interior to P with one endpoint on a vertex of P and the other either at a vertex or on the interior of an edge. Show that in the latter case, α can be chosen so that the endpoints are not on adjacent edges.
(3) Observe that α cuts P into two polygons P_1 and P_2 each with at most $n-1$ sides. Apply the inductive hypothesis.
(4) Prove that if D_1 and D_2 are two spaces homeomorphic to closed discs and if $\alpha_i \subset \partial_i$ is homeomorphic to a closed line segment, then the result of gluing D_1 to D_2 by gluing α_1 to α_2 is still homeomorphic to a disc. Conclude the inductive step of the proof.
(5) Project P onto the sphere using stereographic projection. Rotate the sphere so that the north pole lies interior to P and then project back to $\mathbb{R}^2$. Conclude the proof of the polyognal Schonflies theorem.

2. For Friday

2.1. **Reading Assignment.** Read Section 7.1 and the examples of 7.2. As you read compare the results in this section with our previous results concerning "sequential compactness." Memorize the definition of: open cover, subcover, compact. You may skip the proof of the Heine-Borel theorem: we'll prove it somewhat differently by showing that for metric spaces sequential compactness is equivalent to compactness.

Answer the following questions in an email sent to me by Thursday night at 7 PM with the subject line "HW 7 Friday Reading Assignment." Memorize the definition of

(1) Summarize the proof that the image of a compact space under a continuous map is compact (prop. 4 from 7.1).
(2) List some properties that are equivalent to compactness in at least some instances. Which is easiest for you to understand? Is it always equivalent to compactness?
(3) Summarize the proofs of Propositions 1 and 2 from 7.2.

2.2. **Standard Proofs and Problems (100%).**

(1) From 7.1: problems 1, 3, 6, 12, 13.
(2) From 7.2: Problem 7. This is a proof of Proposition 2, which is an extraordinary useful result.

3. For Monday

Nothing new.