Fall 2018/MA 331 **HW 4: How do you get to Carnegie Hall?**

Before beginning this homework assignment, please review the guidelines for submitting homework. In particular, **If you consult a classmate or online source,** you must give credit for the help you received. Failure to do so may result in a report of academic dishonesty. You are, however, strongly encouraged to work with classmates – just be sure to give them credit for any ideas or help they provide!

Also, please write down the total amount of time spent working on the assignment at the top of what you turn in. If you are spending significantly more than 8 hours per week on homework assignments, you should talk with me to devise a new strategy.

The weekly homework assignments are broken out by day. It is crucial that you meet the deadlines for the reading assignments. When you do the reading, I encourage you to try to prove the theorems/propositions/etc. for yourself before reading the proofs in the book. As you read, sketch additional pictures, make marginal notes. In other words, be an active reader!

For the problems, I strongly encourage you to work with classmates, but be sure you are an active contributor to the discussion. Do not spend time looking for additional online sources. It is easy to waste a lot of time which could be used thinking. There are also a lot of proofs out there which are incorrect or which require a different background from what you have or assume that the course is structured differently.

This homework is different in that there is no additional textbook reading and it is not broken up by day. That doesn't mean you should procrastinate though!

1. Reading Assignment

Please read the handout from the book *Introduction to Topology* by Adams and Franzosa. It concerns applications of topology. Email me responses to the following by the due date of this assignment. Your subject line should be "HW 4 Reading Assignment"

(1) Briefly explain the basic idea of the digital line and digital plane and the relevance of topology.

(2) Briefly explain why a topology, rather than a metric space, is a good model for studying phenotypes.

(3) Briefly explain the relevance of interior and boundary for GIS systems.

2. Problems and Proofs

(1) Pick two problems from HW #2 and either do them for the first time or rewrite them to make them better. Your grades on these problems will affect your HW #2 grade, but not your HW# 4 grade.

(2) A topological space X is **sequentially compact** if whenever (x_n) is a sequence in X, there is a convergent subsequence (x_{n_k}). (Recall that a sequence (a_n) converges to a point $a \in X$ if for every open set $U \subset X$ containing a, the sequence is eventually entirely contained in U. A function $f : X \to Y$ is **sequentially continuous** if whenever (x_n) is a sequence in X converging to some point $a \in X$, then the sequence $(f(x_n))$ in Y converges to $f(a)$.

Some of these problems are similar to what you did with metric spaces, only now they are with topological spaces. They serve to reinforce important concepts.

(a) Prove that in a space with the indiscrete topology, every sequence converges to every point of the space. Thus, indiscrete spaces are sequentially compact.
(b) Prove that infinite sets with the discrete topology are not sequentially compact.
(c) Prove that if X and Y are topological spaces such that X is sequentially compact, and if $f: X \to Y$ is surjective and sequentially continuous then Y is also sequentially compact.
(d) Prove that if X and Y are sequentially compact topological space, then so is $X \times Y$. Use induction to conclude that the subset $[-n,n]^k = [-n,n] \times \cdots \times [-n,n]$ is a sequentially compact subset of $\mathbb{R}^k$.
(e) Proved that a closed subset of a sequentially compact space is sequentially compact. Conclude that every closed and bounded subset of $\mathbb{R}^k$ is sequentially compact.

(3) Determine if the tangent disc topology is 1st countable, 2nd countable, both, or neither.

(4) Prove that there is no homeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ taking the x-axis to the closed unit disc. (Hint: think about interiors, boundaries, sequential compactness, etc.)

(5) Prove that there is no homeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ taking the the open interval $(0,1)$ to the x-axis. (Hint: use compact sets in a creative fashion.) Prove, however, that there is a homeomorphism $h: (0,1) \to (-\infty, \infty)$. (It's just that h can't be extended to a homeomorphism of all of $\mathbb{R}^2$.)

(6) Let $*$ be some point not in $\mathbb{R}^2$. Define a topology on $X = \mathbb{R}^2 \cup \{*\}$ by declaring: every open (with the euclidean topology) set of $\mathbb{R}^2$ is in the topology. Also in the topology are all sets of the form $U \cup \{*\}$ where U is the complement in $\mathbb{R}^2$ of a closed and bounded (with the euclidean topology) set in $\mathbb{R}^2$.

(a) Prove that this defines a topology $\mathscr{T}$ on X.
(b) Is the topology Hausdorff?
(c) Is the topology 2nd countable?
(d) Prove that the topology is sequentially compact.
(e) (Bonus!) Prove that this space is homeomorphic to the unit sphere. (Hint: Use stereographic projection.)

(7) Let X be a topological space and give $X \times X$ the product topology. Define:

$$\Delta: X \to X \times X$$

by $\Delta(x) = (x, x)$. Prove that Δ is continuous.

(8) Define $\sim$ on $\mathbb{R}$ by declaring $x \sim y$ if and only if there exists $n \in \mathbb{Z}$ such that $x - z = n$.

(a) Show that $\sim$ is an equivalence relation.
(b) Prove that the quotient space $\mathbb{R}/\sim$ is homeomorphic to the unit circle.

(9) Define $\sim$ on $\mathbb{R}^2$ by declaring $(x, y) \sim (a, b)$ if and only if there exists $n, m \in \mathbb{Z}$ such that $(x-a, y-b) = (n, m)$.

(a) Show that $\sim$ is an equivalence relation.
(b) Prove that the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to $S^1 \times S^1$. (You may use the fact that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.)