Fall 2018/MA 331 HW 1: Metric Spaces

Before beginning this homework assignment, please review the guidelines for submitting homework. In particular, **If you consult a classmate or online source,** you must give credit for the help you received. Failure to do so may result in a report of academic dishonesty. You are, however, strongly encouraged to work with classmates – just be sure to give them credit for any ideas or help they provide!

Also, please write down the total amount of time spent working on the assignment at the top of what you turn in. If you are spending significantly more than 8 hours per week on homework assignments, you should talk with me to devise a new strategy.

The weekly homework assignments are broken out by day. It is crucial that you meet the deadlines for the reading assignments. When you do the reading, I encourage you to try to prove the theorems/propositions/etc. for yourself before reading the proofs in the book. As you read, sketch additional pictures, make marginal notes. In other words, be an active reader!

For the problems, I strongly encourage you to work with classmates, but be sure you are an active contributor to the discussion. Do not spend time looking for additional online sources. It is easy to waste a lot of time which could be used thinking. There are also a lot of proofs out there which are incorrect or which require a different background from what you have or assume that the course is structured differently.

In this (and subsequent assignments) you will need to know the concepts of "infimum" and "supremum." Given a subset $A \subset \mathbb{R}$, a **lower bound** for A is any $\alpha \in \mathbb{R}$ such that for all $a \in A$

$$\alpha \leq a.$$

If $A = \varnothing$, we set $\inf A = \infty$. If A does not have a lower bound then we set $\inf A = -\infty$. If A is non-empty and has a lower bound, we let $\inf A$ be the "greatest lower bound." That is, $\inf A$ is the real number such that

$$\alpha \leq \inf A$$

for every lower bound α of A. It is an important consequence of a formal definition of real numbers that every non-empty subset $A \subset \mathbb{R}$ which has a lower bound also has a unique **infimum** $\inf A$. Some authors use "greatest lower bound" instead of infimum and "glb" instead of inf.

Similarly, an **upper bound** for A is a real number β such that for all $a \in A$,

$$a \leq \beta.$$

If $A = \varnothing$, then $\sup A = -\infty$. If A does not have an upper bound, then $\sup A = \infty$. Otherwise, if A is non-empty and has an upper bound, $\sup A$ is the "least upper bound." That is, the real number $\sup A$ such that

$$\sup A \leq \beta$$

for every upper bound β of A.

The key property of the real numbers is:

Theorem. *If $A \subset \mathbb{R}$ then A has both an infimum and a supremum (possibly $\pm\infty$).*

In practice, we use the following properties of infima and supremum.

- Suppose that $A \subset \mathbb{R}$ is non-empty and has a lower bound (so $\inf A$ is a real number and not $\pm\infty$). Then for every $\epsilon > 0$, there exists $a \in A$ such that
$$\inf A \le a \le \inf A + \epsilon$$
In other words, moving a little bit to the right of $\inf A$ on the number line, we can capture some element of A in the interval $[\inf A, \inf A + \epsilon]$.
- Suppose that $A \subset \mathbb{R}$ is non-empty and has an upper bound (so $\sup A$ is a real number and not $\pm\infty$). Then for every $\epsilon > 0$, there exists $a \in A$ such that
$$\sup A - \epsilon \le a \le \sup A.$$
In other words, moving a little bit to the left of $\sup A$, we can capture some element of A in the interval $[\sup A - \epsilon, \sup A]$.

1. For Friday

1.1. Reading.

1.1. Reading. From the textbook, read Section 2.1. You do not need to study the geometric proof of the inequality marked ⧫ on page 18. You should study the proof of Proposition 1. Perhaps try to write your own proof of it before reading the one in the book.

Email me answers to the following questions by 7 PM on Thursday. The subject line of your email should be "MA 331: HW 1 Friday Reading Assignment"

(1) Which of the concepts from this section have you seen before? What are new?
(2) Which terms seem like they will be the most important?
(3) There are several explanations of facts, propositions, and examples in this section. Which explanations did you understand easily? which did you understand after some effort? which did you not understand at all?
(4) How much time did it take you to do the reading and answer the previous questions?

1.2. Exercises. (85 %)

Exercises can generally be done by carefully putting together definitions or previous results.

(1) Suppose that X is a non-empty set and that d is the discrete metric on X. Prove that for every $x \in X$, the set $\{x\}$ is open.

(2) Do Problems 1 and 6 from Section 2.1.

(3) Suppose that X is a metric space with metric d. Let $Y \subset X$. For $y_1, y_2 \in Y$, define $d_Y(y_1, y_2) = d(y_1, y_2)$. (The function d_Y is called the **restriction** of d to the subset Y.) Explain why d_Y is a metric on Y. (It is called the **subspace metric** on Y inherited from X.)

(4) Spell out the proof of Proposition 2 on page 22.

1.3. Advanced Problems. (15 %)

Advanced problems require significant thought and creativity. You are not necessarily intended to solve them all. If you work on a problem, but don't solve it, you should explain what you've tried and how far you've gotten.

(1) A **path** in the plane $\mathbb{R}^2$ (with the euclidean metric) is a continuous, piecewise smooth function $\gamma = \gamma(t) = (x(t), y(t))$ where the domain of γ consists of all t in some interval $[a, b] \subset \mathbb{R}$. (For

instance, the path $\gamma(t) = (t, t^2)$ for $t \in [-1, 1]$ traces out a portion of a parabola in $\mathbb{R}^2$.) We say that the path is a path from the point $\gamma(a) \in \mathbb{R}^2$ to the point $\gamma(b) \in \mathbb{R}^2$

In Calculus, the **length** of such a path γ is defined to be:

$$\ell(\gamma) = \int_a^b ||\dot{\gamma}(t)||\, dt = \int_a^b \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt.$$

Given a subset U, we will use path length to define a metric, called the **path metric** on U.

Suppose that $U \subset \mathbb{R}^2$ is a non-empty subset such that for all $u_1, u_2 \in U$ there is a path γ from u_1 to u_2 and $\gamma(t) \in U$ for all t. We say that any such γ is a path **in** U. For $u_1, u_2 \in U$, define

$$d_P(u_1, u_2) = \inf\{\ell(\gamma) : \gamma \text{ is a path in } U \text{ from } u_1 \text{ to } u_2.\}$$

(a) Prove that d_P is a metric on U.

(b) Give an example of a subset $U \subset \mathbb{R}^2$ for which d_P is defined but is not equal to the euclidean metric d. (Hint: Convex subsets of $\mathbb{R}^2$ have the property that the path metric is equal to the euclidean metric.

(2) Do problems 5, 8, 9, and 10. There is a typo in the statement of #10. It should say $G \subset V$.

2. For Monday

2.1. **Reading.** Read Section 2.2 up through the proof of Proposition 5. We will go over the proof of Theorem 2 (the Nested Interval Theorem) in class. You should memorize the following definitions:

- sequence converging to a point.
- Cauchy sequence
- complete metric space
- $f : X \to Y$ between metric spaces is continuous at $x \in X$.
- homeomorphism
- product metric
- uniformly continuous

Study the proofs of Example 1, Proposition 3 and Proposition 4. You may like to try to come up with your own proofs for those propositions before reading the proofs in the book. By 8 PM on Sunday, email me answers to the following questions. Your email should have the subject line: "MA 331: HW 1 Monday Reading Assignment"

(1) Which definitions did you struggle to understand? Which don't you understand?
(2) Which proofs are you still unclear on?

2.2. **Exercises.** (85%) Exercises can generally be done by carefully putting together definitions or previous results. The following exercises are from Section 2.2.

(1) Exercises 1, 2 (Hint: show it converges to the supremum of the range of the sequence), 5 (this one is important), 8, 14.

Hints:

(5, ⇒) Assume that $A \subset X$ is closed. Show that every sequence which converges in X also converges in A. Let (a_n) be a sequence in A converging to $a \in X$. We must show that $a \in A$. Assume, for a contradiction, that $a \notin A$. Use the fact that A is closed, to show that there is some $r > 0$ such that $B(a, r)$ is disjoint from A. Use this to contradict the assumption that $(a_n) \to a$.

(5, $\Leftarrow$) Prove the contrapositive. Assume that A is not closed and show that it has a sequence converging to some element of $X \setminus A$. Since A is not closed, its complement is not open. Thus, there exists $x \in X \setminus A$ such that for all $r > 0$, $\langle$*finish the proof*$\rangle$.

2.3. **Advanced Problems.** (15 %) Advanced problems usually require significant thought and creativity. You are not necessarily intended to solve them all. If you work on a problem, but don't solve it, you should explain what you've tried and how far you've gotten.

A metric space (X, d) is said to be **sequentially compact** if it has the following property: whenever (x_n) is a sequence in X, it has a convergent subsequence (x_{n_k}).

(1) Suppose that (x_n) is a sequence in a closed, bounded interval $[a, b] \subset \mathbb{R}$. Prove: (x_n) has a subsequence which is increasing or (x_n) has a subsequence which is decreasing. (Hint: Adapt part of the proof of Example 1 on page 25.)

(2) Prove that $[a, b]$ is sequentially compact.

(3) Suppose that X is a sequentially compact metric space and that $A \subset X$ is closed. Prove that A (with the subspace metric) is sequentially compact.

(4) Suppose that (X, d_X) and (Y, d_Y) are sequentially compact metric spaces. Give $X \times Y$ the product metric d (denoted d_P on page 27). Prove that $X \times Y$ is sequentially compact.

(5) Suppose that (X, d_X) and (Y, d_Y) are metric spaces with X sequentially compact. Suppose that there exists a surjective continuous function $f \colon X \to Y$. Prove that Y is sequentially compact.