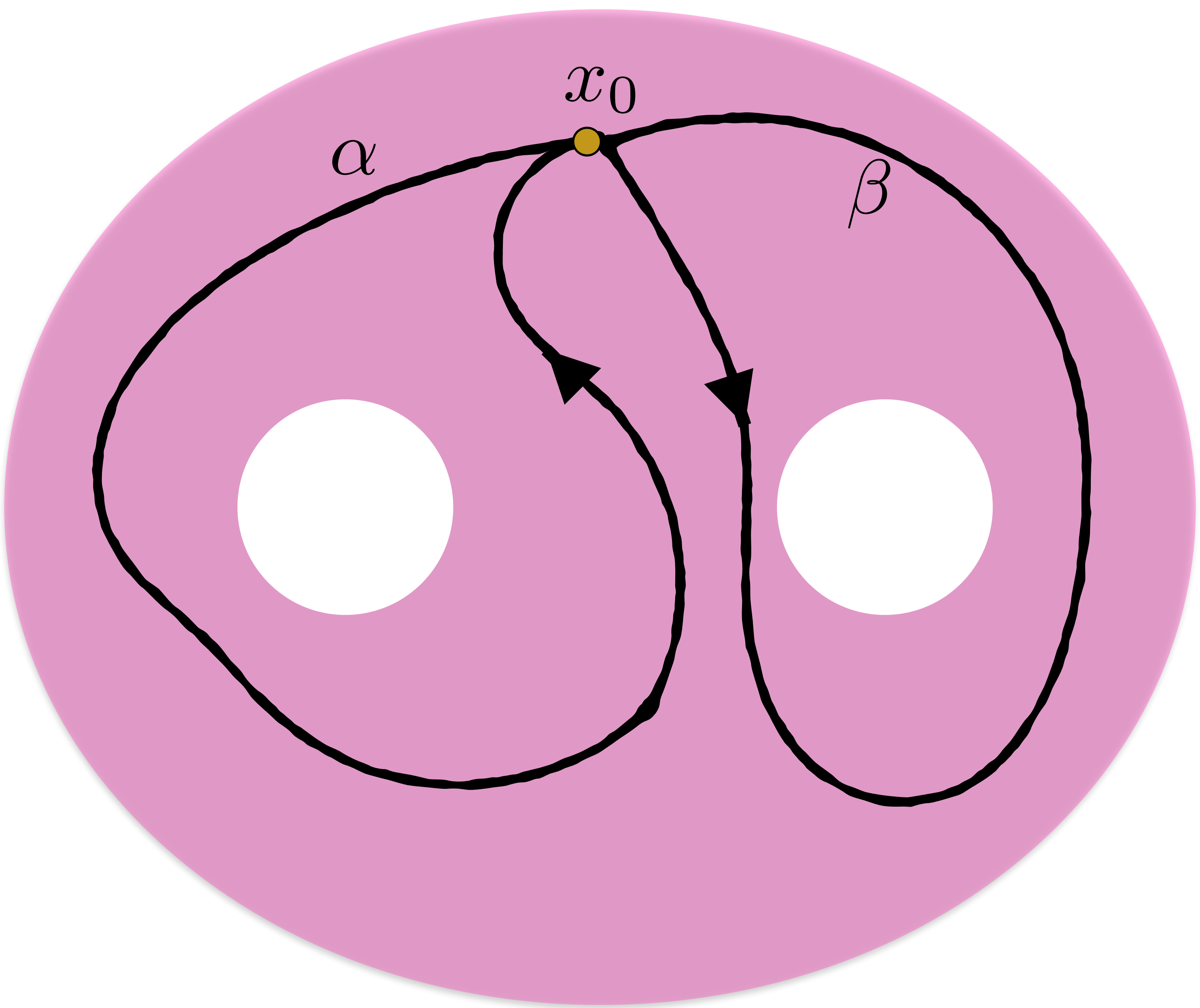
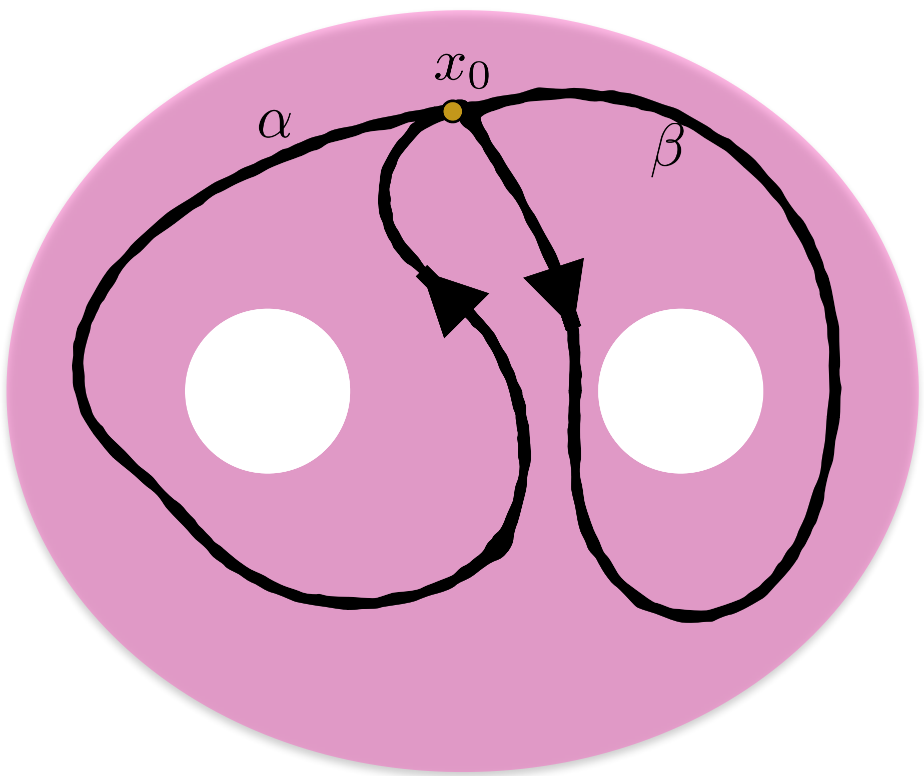
# The groupiness of $\pi_1(X, x_0)$

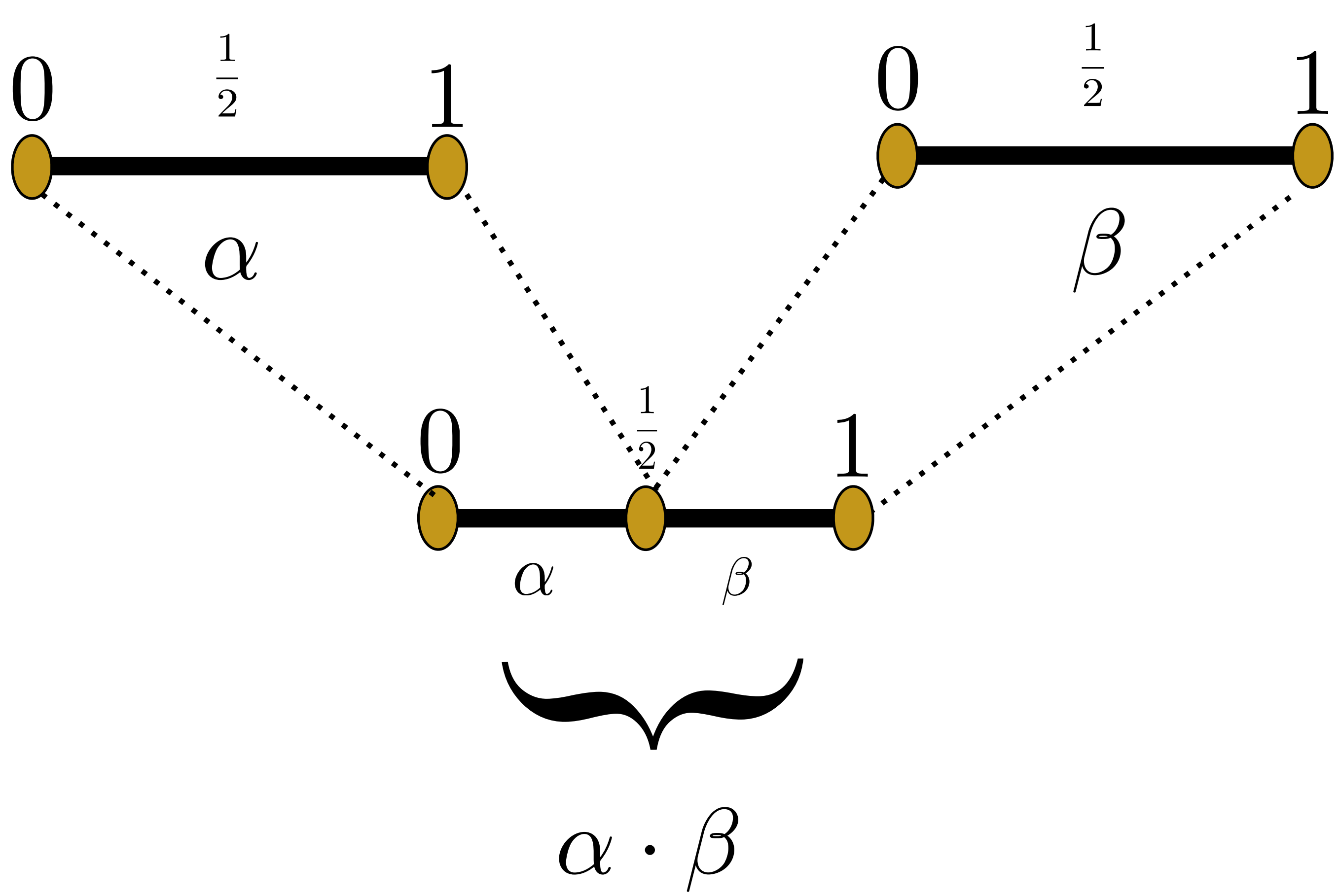


# Concatenation

$$\alpha: [0,1] \to X$$
$$\beta: [0,1] \to X$$
$$\alpha \cdot \beta: [0,1] \to X$$

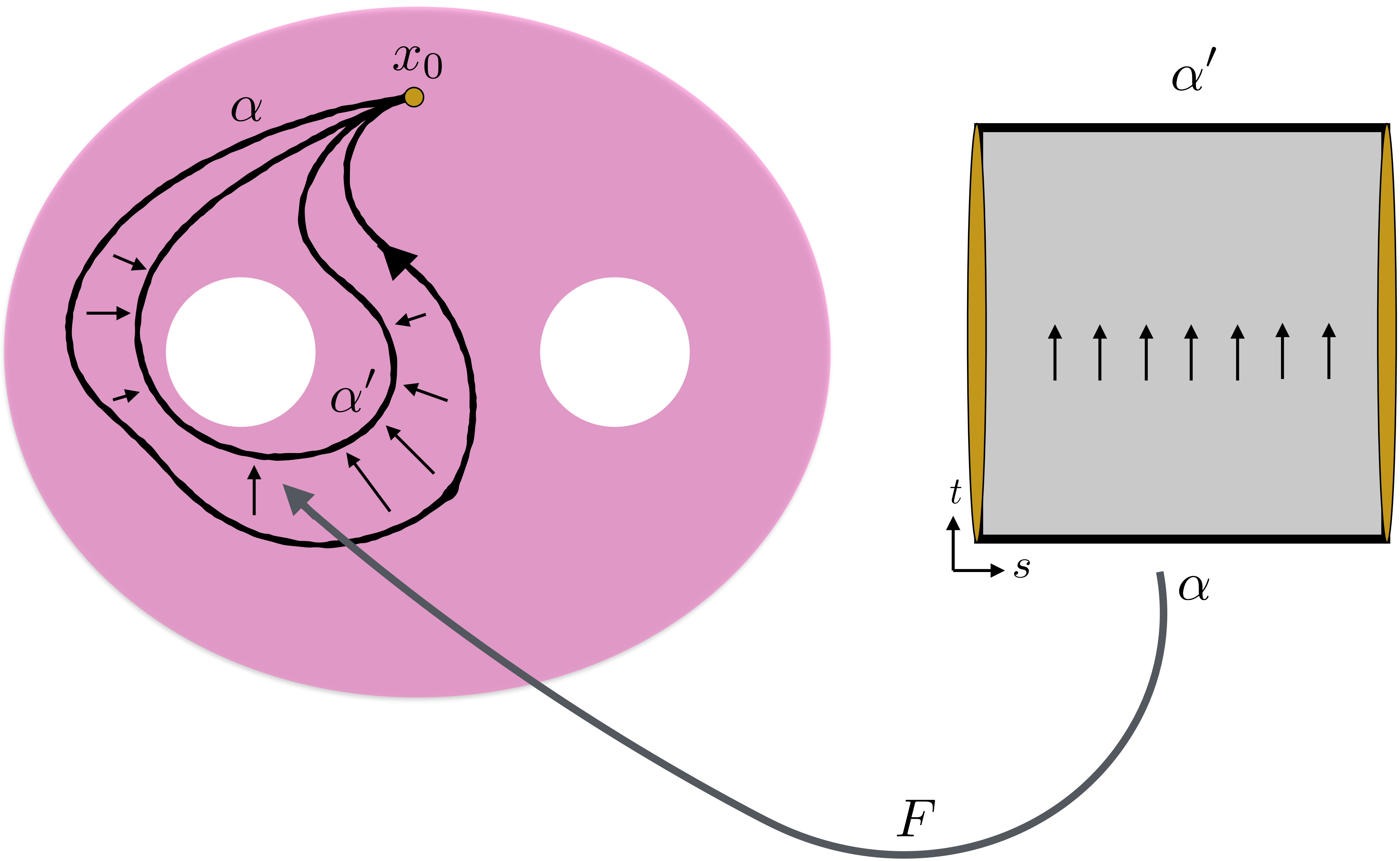



$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ \beta(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

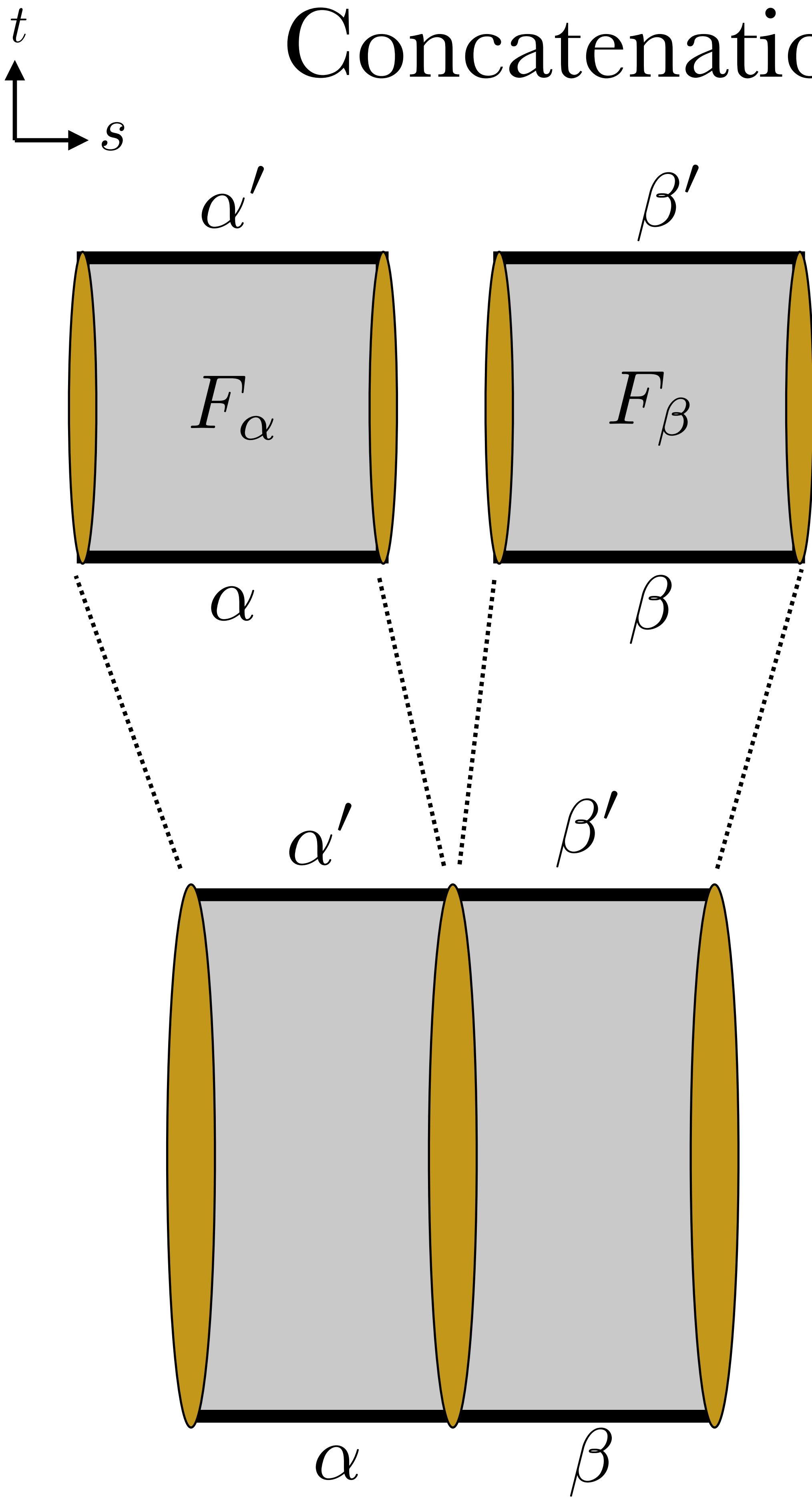


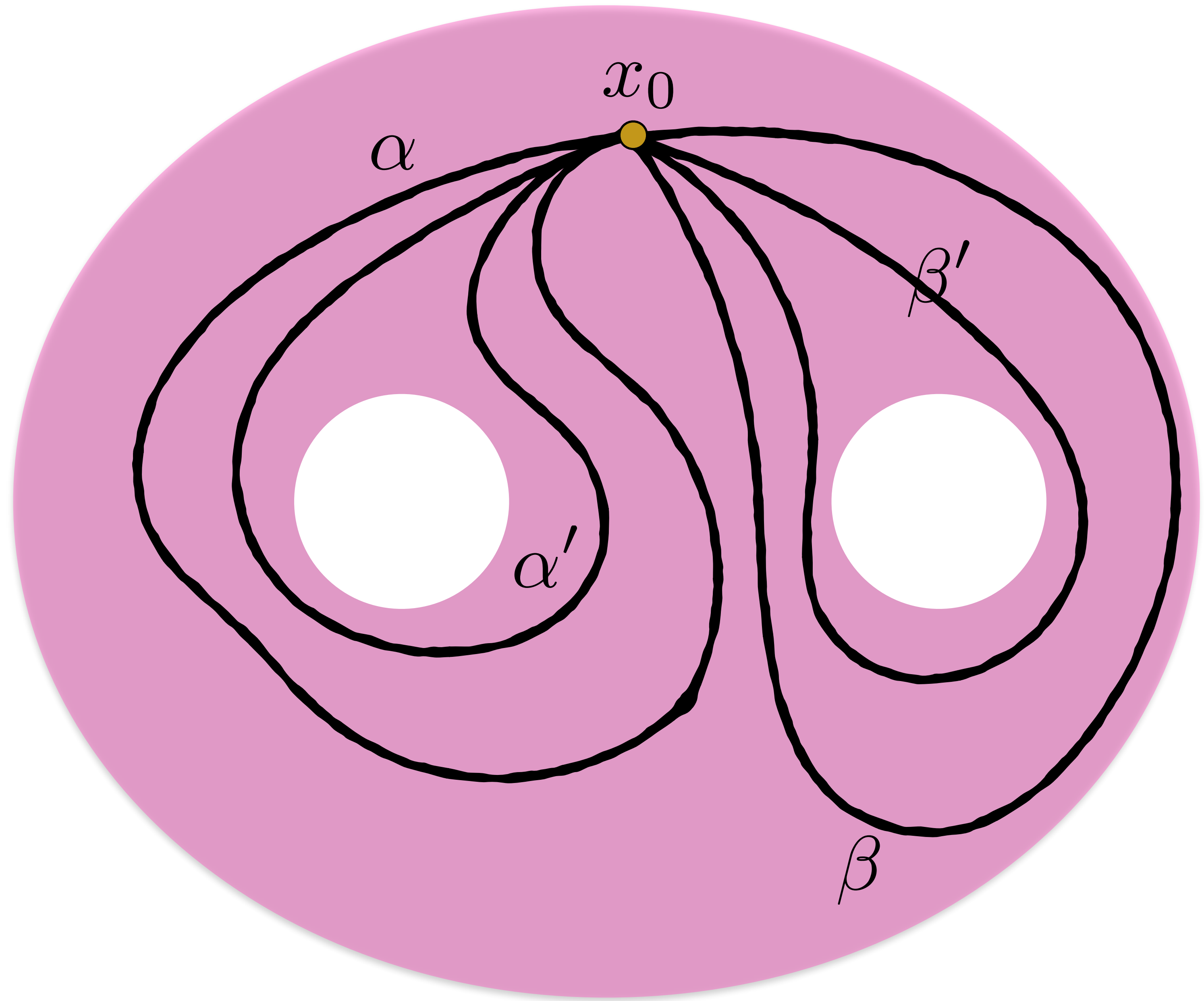
# Basepoint preserving homotopy

$F : [0,1] \times [0,1] \to X$



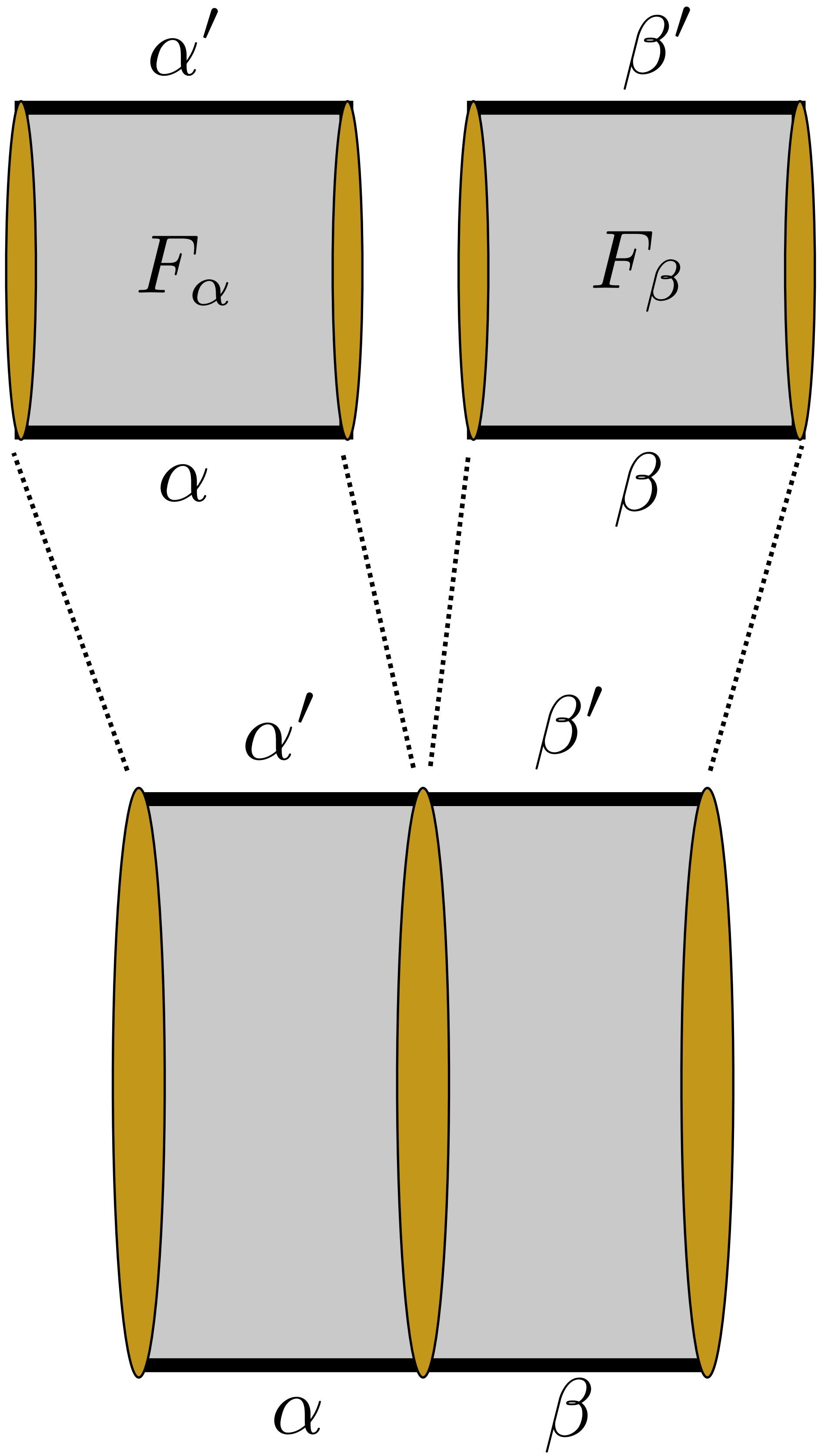
# Concatenation and b.p. homotopy




$$F(s,t) = \begin{cases} F_\alpha(2s,t) & 0 \le s \le 1/2 \\ F_\beta(s,t) & 1/2 \le s \le 1 \end{cases}$$

# Concatenation and b.p. homotopy




Write $\alpha \simeq \alpha'$ to mean

$\alpha$ is b.p. homotopic to $\alpha'$.

And let

$$[\alpha] = \{\alpha' : \alpha \simeq \alpha'\}$$

be the equivalence class.

We have shown:

$$[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$$

is a well-defined binary operation on the set of equivalence classes, denoted: $\pi_1(X, x_0)$

# A group consists of a:

- set: G
- associative binary operation
- identity element $\mathbf{1}$ s.t. $\mathbf{1} \cdot a = a = a \cdot \mathbf{1}$
- inverse for each element s.t.

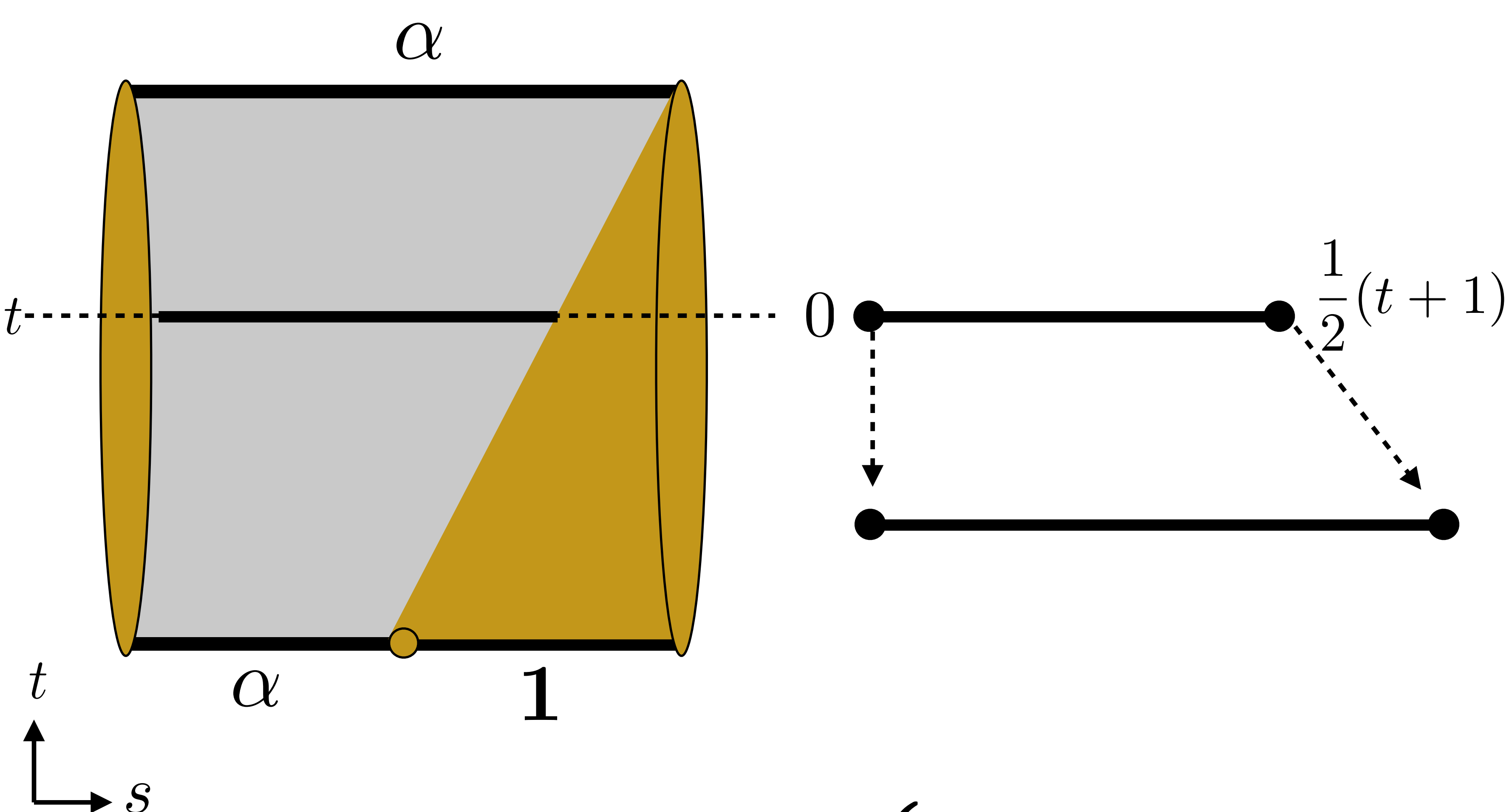
$$a^{-1} \cdot a = \mathbf{1} = a \cdot a^{-1}$$

# The identity

Let $\mathbf{1}(s) = x_0$ for all $s \in [0,1]$.

<u>Claim:</u> $[\alpha] \cdot [\mathbf{1}] = [\alpha] = [\mathbf{1}] \cdot [\alpha]$ for all $[\alpha] \in \pi_1(X, x_0)$.

That is, $\alpha \cdot \mathbf{1} \simeq \alpha \simeq \mathbf{1} \cdot \alpha$ for all $[\alpha] \in \pi_1(X, x_0)$.




$$\phi_t(s) = \frac{2s}{t+1}$$

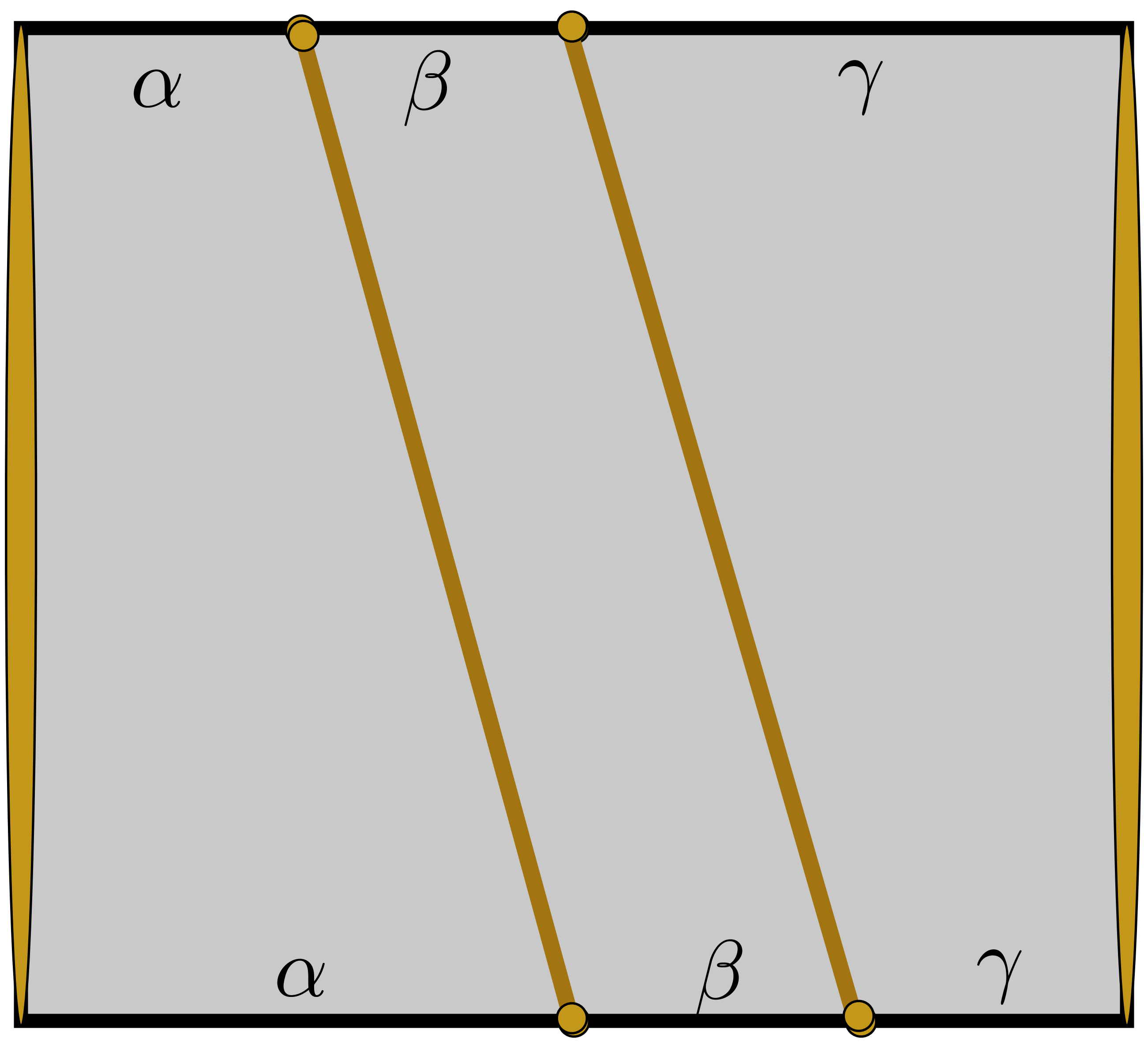
$$F(s,t) = \begin{cases} \alpha(\phi_t(s)) & 0 \le s \le (t+1)/2 \\ x_0 & (t+1)/2 \le s \le 1 \end{cases}$$

# Associativity

We must show: $[\alpha] \cdot ([\beta] \cdot [\gamma]) = ([\alpha] \cdot [\beta]) \cdot [\gamma]$

for all $[\alpha], [\beta], [\gamma] \in \pi_1(X, x_0)$.

That is: $\alpha \cdot (\beta \cdot \gamma) \simeq (\alpha \cdot \beta) \cdot \gamma \quad \forall \alpha, \beta, \gamma$

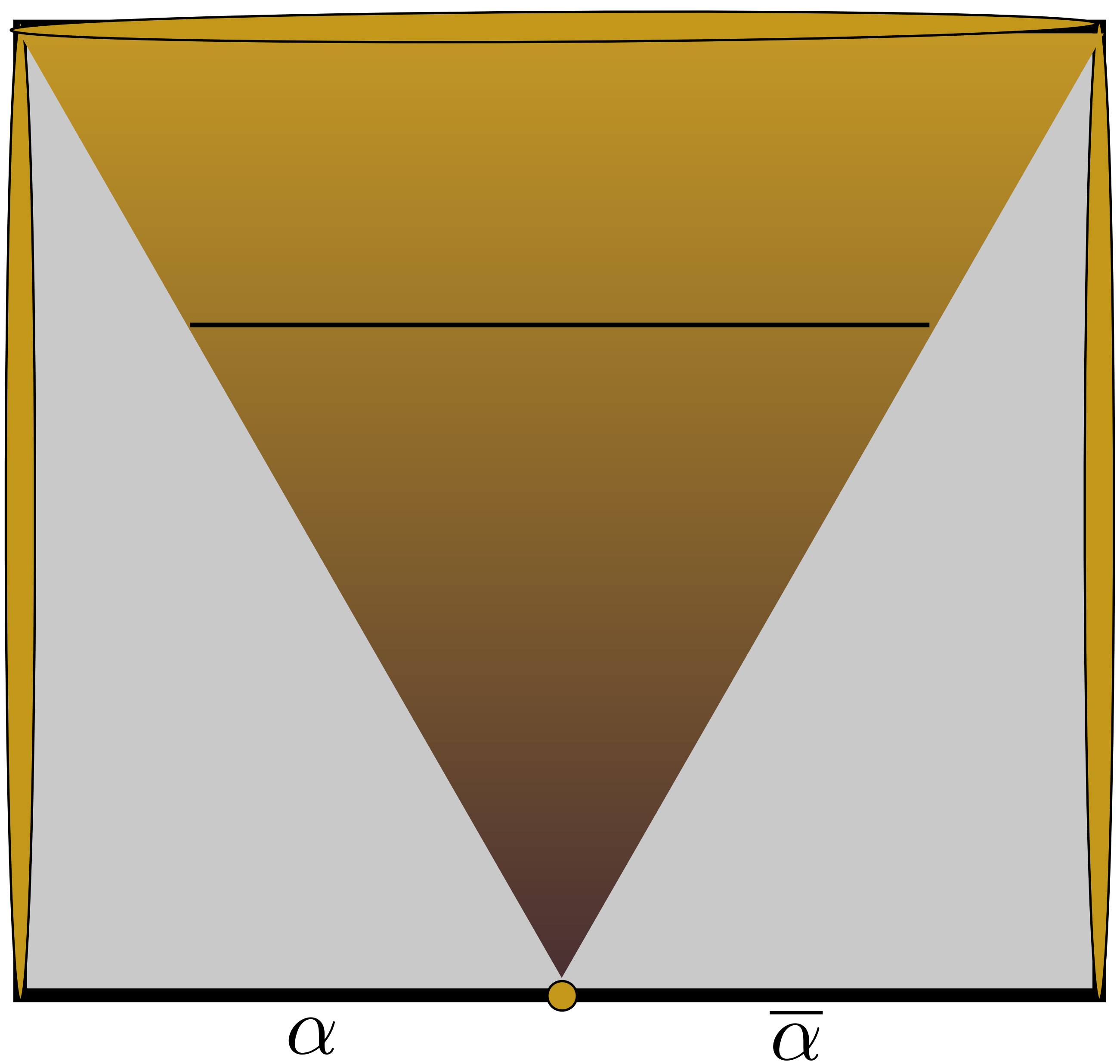


# Inverses

Recall the reverse of $\alpha$ is $\overline{\alpha}$,

$$\overline{\alpha}(s) = \alpha(1-s) \text{ for all } s \in [0,1].$$

We'll show: $\quad \overline{\alpha} \cdot \alpha \simeq \mathbf{1} \simeq \alpha \cdot \overline{\alpha}$




On each horizontal line, in the triangle we remain constant.

$$F(s,t) = \begin{cases} \alpha(s) & 0 \le s \le (1-t)/2 \\ \alpha((1-t)/2) & (1-t)/2 \le s \le (t+1)/2 \\ \overline{\alpha}(s) & (t+1)/2 \le s \le 1 \end{cases}$$