Fall 2018/MA 331 Exam 1 Study Guide

The exam will consist of a section of definitions, a section of old problems (eg. from Homework or class), and at least one new problem. Here is a list of problems (mostly ones you've seen before) to study. You should be pro-active about finding others to study, too.

(1) Let X be a set having the discrete metric. Prove that every subset of X is open, when using the topology induced by the discrete metric.

(2) Let X be a set with a metric d. Prove that every singleton $\{x\}$ is closed.

(3) Let d_1 and d_2 be metrics on a set X. Define d on X by $d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$. Prove that d is a metric.

(4) Prove that in a metric space (X,d), the uinon of any number of open sets is open.

(5) Prove Cantor's nested interval theorem and use it to prove the Cantor set has uncountably many elements.

(6) Give an outline of a proof of the Intermediate Value Theorem.

(7) Prove that a sequence in a metric space can have at most one limit.

(8) Prove that a bounded, monotonic sequence in $\mathbb{R}$ converges.

(9) If (x_n) is a Cauchy sequence in a metric space such that it has a convergent subsequence, then (x_n) also converges.

(10) Let (X,d_1) and (Y,d_2) be metric spaces. Suppose that (a_n,b_n) is a sequence in $X\times Y$. Assuming $X\times Y$ has the product metric, prove that (a_n,b_n) converges if and only if (a_n) converges and (b_n) converges.

(11) Prove that a subset A of a metric space X is closed if and only if every sequence (a_n) in A which converges in X also converges in A.

(12) Suppose that X is a set and Y is a topological space. Prove that $\{f^{-1}(V) : V \subset Y \text{ is open }\}$ is a topology on X.

(13) Let X be a set and $x \in X$. Let $\tau \subset \mathscr{P}(X)$ be the set whose elements are precisely the sets containing x. Prove that $\tau \cup \{\varnothing\}$ is a topology on X.

(14) Prove that the intersection of topologies is a topology.

(15) Suppose that $f : X \to Y$ and that X is a topological space. Prove that $\{U \subset Y : f^{-1}(U) \text{ is open in } X\}$ is a topology on Y.

(16) A subset A of a topological space X is closed if and only if it contains all its accumulation points.

(17) Suppose that A is a subset of a topological space X. Let $\overline{A}$ be the smallest closed set in X containing A. Prove that $\overline{A} = A \cup A'$.

(18) Problem 17 from Section 3.2.

(19) Show the Sorgenfrey topology on $\mathbb{R}$ is a superset of the euclidean topology. (i.e. it is finer)

(20) Show that $\mathbb{R}^n$ with the usual topology is second countable.

(21) Prove the wedge of countably many circles is not homeomorphic to any subspace of $\mathbb{R}^2$.

(22) Proposition 5 from Section 3.5.

(23) Show that every mapping from a discrete space is continuous, as is any mapping to an indiscrete space.

(24) Prove that the composition of continuous functions is continuous.

(25) Find topological spaces X and Y and a continous function $f\colon X \to Y$ which is open, but not closed.

(26) Exercise 9 from Section 3.5.

(27) Suppose that Z is a topological space and that $Y \subset Z$ has the subspace topology. Given $X \subset Y$, we can give X the subspace topology τ_1 from Y or the subspace topology τ_2 from Z. Prove $\tau_1 = \tau_2$.

(28) Suppose that X is a topological space and let $A \subset X$. Prove that a subset $B \subset A$ which is open in A is also open in X if A is open in X.

(29) Suppose that X is a sequentially continuous topological space and that Y is a topological space. If $f\colon X \to Y$ is a surjective continous function, then Y is also sequentially continuous.

(30) Let $Z = [0,1]/\{0,1\}$. Prove that Z is homeomorphic to S^1.

(31) Let $\mathbb{Z}$ act on $\mathbb{R}$ by defining $n \cdot x = n + x$. Prove that this defines a group action and that the quotient space $\mathbb{R}/\mathbb{Z}$ by this group action is homeomorphic to S^1.

(32) Suppose that X and Y are topological spaces and that $X \times Y$ has the product topology. Let $y_0 \in Y$. Prove that $\iota\colon X \to X \times Y$ defined by $\iota(x) = (x, y_0)$. Prove that ι is continuous.

(33) State and prove the gluing lemma.

(34) Give an outline of the proof that for the Cantor set C, the space $C \times C$ is homeomorphic to C.