Fall 2018/MA 274 GP 1: Groups

Work through this sheet with your group (if time runs out, you do not need to complete it) and then write a one page summary of your findings to turn in. (Each person should write their own summary.)

Abstraction plays an important role in advanced mathematics. When we study an abstract mathematical structure we better understand what precisely makes certain facts true or false and we derive conclusions which are valid in a wide variety of particular situations. In this project, you'll explore the notion of a "group". This is an abstract concept which encapsulates important properties of objects you know (and hopefully love!)

A **group** consists of a set (i.e. a collection) of numbers, functions, or other mathematical objects along with a way of combining any two of the objects to get a third. For example, the following are potentially groups:

(1) The natural numbers $\mathbb{N} = \{1,2,3,4,\ldots\}$ along with addition $+$ as the method of combination.

(2) The natural numbers $\mathbb{N} = \{1,2,3,4,\ldots\}$ along with multiplcation $\cdot$ as the method of combination.

(3) The integers $\mathbb{Z} = \{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\}$ along with addition $+$ as the method of combination.

(4) The integers $\mathbb{Z} = \{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\}$ along with multiplcation $\cdot$ as the method of combination.

(5) The integers $\mathbb{Z} = \{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\}$ along with subtraction $-$ as the method of combination.

(6) The rational numbers $\mathbb{Q}$ (these are the set of all fractions a/b of integers, but requiring $b \neq 0$) with $+$ as the method of combination

(7) The rational numbers $\mathbb{Q}$ with multiplication $\cdot$ as the method of combination.

(8) The rational numbers $\mathbb{Q}$, but excluding 0, with division as the method of combination.

(9) The real numbers $\mathbb{R}$ (i.e. all the numbers on the number line) with multiplication as the operation.

(10) The real numbers $\mathbb{R}$ but excluding 0 with multiplication as the operation.

(11) The functions $f(x) = x$, $g(x) = 1/x$ with composition $\circ$ as the operation.

(12) All functions f with domain and range the real numbers.

(13) All transformations of the Cartesian plane $\mathbb{R}^2$ which don't change the distance between any two points.

1. Go through the previous examples and convince yourself that if you use the method of combination on any two of the items in the set, you'll get something else in the set. Are there any you have difficulty with?

To discuss any one of these set-ups (as well as many others) it helps to have notation which could refer to any one of them. We'll call the set G and the method of combining $\circ$. (So depending on the situation, $\circ$ might mean the same thing as $+$ or $\cdot$ or function composition or something else.) To be a **group** the set G and the operation $\circ$ must satisfy the following requirements (called **axioms**):

(G1) (closure) For each pair of elements a and b in G, there is a unique element c in G such that $c = a \circ b$.

(G2) (identity) There is at least one element (which we'll denote $\mathbb{1}$) in G such that for every element a in G:

$$a \circ \mathbb{1} = a \text{ and } \mathbb{1} \circ a = a.$$

(G3) (inverses) For every element a in G there is at least one element (which we'll denote a^{-1}) in G such that

$$a \circ a^{-1} = \mathbb{1} \text{ and } a^{-1} \circ a = \mathbb{1}.$$

(G4) (associative) For every choice a, b, and c of elements in G, we have:

$$a \circ (b \circ c) = (a \circ b) \circ c.$$

2. For each of the previous examples of a set and a method of combination, try to figure out which are groups and which aren't. For each group axiom, you should do your best to explain either why it is true or why it isn't. At this point of the semester, your explanations will likely leave a lot to be desired. That's totally fine! Just do your best! If you can't really come up with an answer, explain something of your thought process.

3. Notice that in axiom (G2) it says that there is at least one element with that desired property. Could there be more than one? Suppose that there are two such elements, call them $\mathbb{1}$ and $\mathbb{1}'$ satisfying the two equations in (G2). Using only the axioms (G1), (G2), (G3), and (G4) show that $\mathbb{1} = \mathbb{1}'$.

4. Suppose we have a group G with operation $\circ$. Suppose that a is some element of G. Notice that in axiom (G2) it says that there is at least one element (denoted a^{-1}) with that desired property. Could there be more than one? Suppose that there are two such elements, call them g and g' satisfying the two equations in (G2) (in place of a^{-1}). Using only the axioms (G1), (G2), (G3), and (G4) show that $g = g'$.