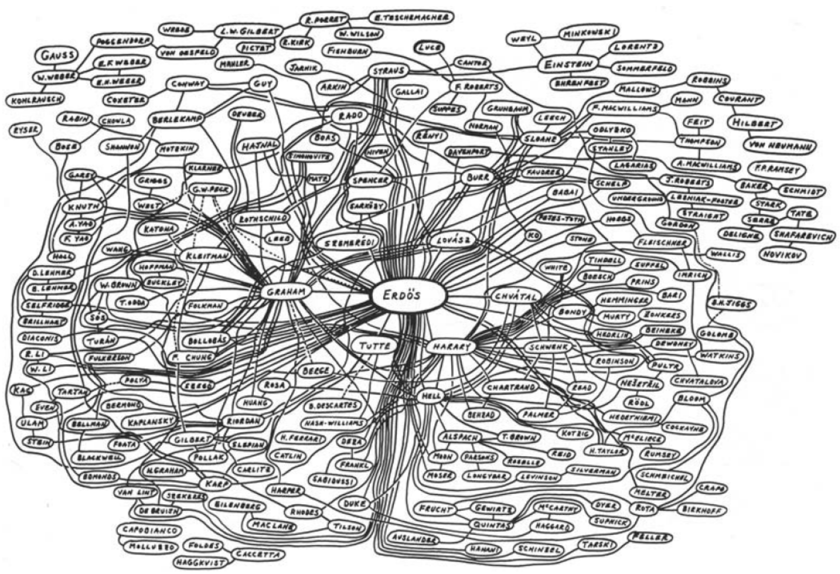
Fall 2018/MA 274 GP 2: Graphs

Work through this sheet with your group (if time runs out, you do not need to complete it) and then write a one page summary of your findings to turn in. (Each person should write their own summary.) As part of your summary you should say who your group members were and discuss the contributions of each (including yourself).

The word "graph" in mathematics has multiple meanings. One meaning is the "graph of a function," as in Calculus. Another meaning, and the one we'll use for this project, is that of a network, consisting of nodes and connections between the nodes. For example, there is the Facebook graph where each node (or **vertex**) is a Facebook user and there is a connection between them if they are "friends." Similarly, there is the collaboration graph in mathematics. Here is a small portion of it, involving the famous mathematician Paul Erdös. (The image is from the blog "Not quite the economist" and was hand-drawn by the graph theorist Ron Graham.) Each node is a mathematician and two nodes are joined if the mathematicians co-authored a paper.



The purpose of this project is to study ways of moving through graphs. Once again at this point in the semester you won't be able to provide perfect answers! Don't worry about it, just do your best.

(1) Figure 1 shows a way of moving through a graph along a path.

To specify our route, we can list all the vertices we pass through:

$$a, v_1, v_2, v_3, v_4, v_5, b$$

The vertex v_3 is the same as the vertex v_5, so there are vertices that appear more than once in the list.

Draw your own example of a fairly complicated graph and specify a path in it between two vertices.

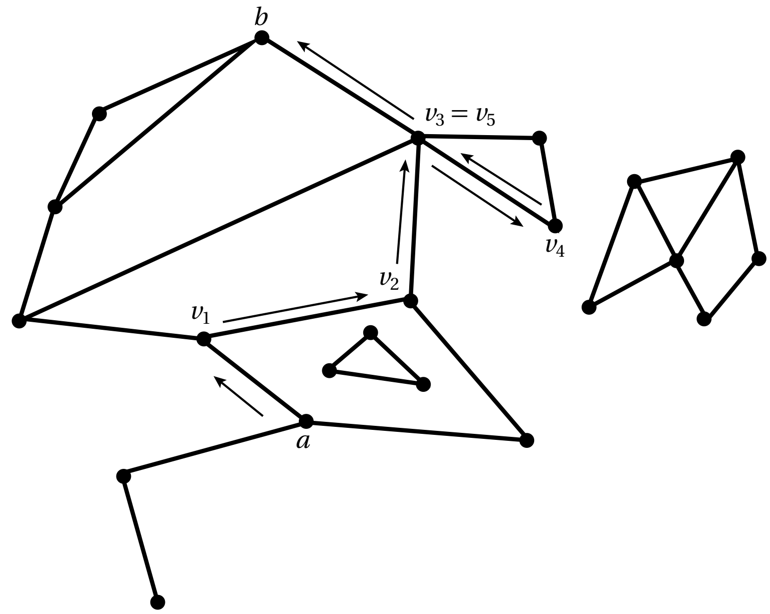


FIGURE 1. The finite sequence $a, v_1, v_2, v_3, v_4, v_5, b$ is a path from a to b of length 6.

(2) Here is the formal definition of a path in a graph:

Definition. Suppose that G is a graph with vertices V and edges E. A **path** in G is a finite list of vertices

$$\alpha = v_0, v_1, \ldots, v_n$$

for some $n \in \{0, 1, 2, 3, 4, \ldots\}$ such that the following conditions hold:

- Each v_i is a vertex of G.
- For all i (with $0 \le i \le n-1$) the vertices v_i and v_{i+1} are the endpoints of an edge.

The number n is the **length** of the path α.

If e is an edge of G such that there are vertices v_i and v_{i+1} which are the endpoints of G, then we say that e is **traversed** by α. If a and b are vertices of G such that $v_0 = a$ and $v_n = b$, we say that α is a path **from** a **to** b.

Warning: If we are discussing more than one path at a time, we won't want to use $v_0, v_1, \ldots, v_n$ to indicate both of them. We could call one $v_0, v_1, v_2, \ldots, v_n$ and the other $w_0, w_1, w_2, \ldots, w_m$, for example.

Explain how the formal definition applies to the example given above and to the example you created. In particular, what is n?

(3) Explain why the following are true. Your explanations should appeal directly to the formal definition of path.

- If there is a path from vertex a to vertex b, then there is also a path from b to a. (Hint: Suppose that $v_0, v_1, v_2, \ldots, v_n$ is a path from a to b. What list of vertices would be a path from b to a?)
- If there is a path from vertex a to vertex b and also a path from vertex b to vertex c, then there is a path from vertex a to vertex c.
- For every vertex a there is always a path from a to itself. (Hint: Explain why a path can consist of a single vertex.)

(4) A graph is **connected** if for every two vertices a and b, there is a path from a to b. For a connected graph G, and for vertices a and b define $d(a,b)$ to be the length of the shortest path from a to b. Prove the following:

(a) For every vertex a, $d(a,a) = 0$.
(b) For all vertices a and b, we have $d(a,b) = d(b,a)$. (Hint: Suppose you have the shortest path from a to b. How can you get the shortest path from b to a?)
(c) For all vertices a, b, c, we have $d(a,c) \leq d(a,b) + d(b,c)$.