MA 274: Exam 2 Study Guide

Here are some suggestions for what and how to study:

Theorems marked (CHALLENGE) are particularly challenging and would placed on the exam only as a challenge problem.

(1) Know the definitions on the website. Any other definitions that you need will be given to you.

(2) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.

(3) Know the theorems we've proved in class and the more significant theorems from the homework.

(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.

(5) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.

(6) Here are some results you should be especially sure to know how to prove; some of them may be new. You should also think about ways these problems might be varied.

(7) Go back and look at the study guide for Exam 1 by way of reviewing the basics. The material from that exam will not be explicitly tested in Exam 2, but of course it forms the basis for what we do.

Here are some sample problems.

(1) The following are examples of equivalence relations:

- $\equiv_p$ on $\mathbb{Z}$, where $x \equiv_p y$ iff $x - y$ is a multiple of p.
- $\sim$ on the vertex set of a graph G, where $x \sim y$ iff there is a path from x to y in the graph.
- Suppose that P is a partition of a non-empty set X. Define $\sim$ on X by $x \sim y$ iff there exists $A \in P$ such that x, y are both elements of A.

(2) Suppose that $\sim$ is an equivalence relation on a non-empty set X. For each $x \in X$, let $[x]$ be the equivalence class of x. Then the following hold:

(a) For all $x \in X$, $x \in [x]$.
(b) For all $x, y \in X$, $x \sim y$ iff $[x] = [y]$.
(c) For all $x, y \in X$, $[x] \cap [y] \neq \varnothing$ implies $[x] = [y]$.

(3) If $\sim$ is an equivalence relation on a non-empty set X, then the set of equivalence classes is a partition of X.

(4) If $f\colon \mathbb{Z}/\equiv_p \to \mathbb{Z}/\equiv_p$ is defined by $f([x]) = [2x]$ then f is well-defined.

(5) Addition and multiplication in $\mathbb{Z}/\equiv_p$ are well-defined.

(6) The compositions of injections/surjections/bijections is a an injection/surjection/bijection.

(7) A function $f\colon X \to Y$ is a bijection if and only if there is a function $f^{-1}\colon X \to Y$ such that $f \circ f^{-1}(y) = y$ for all $y \in Y$ and $f^{-1} \circ f(x) = x$ for all $x \in X$.

(8) The set of bijections from a set X to itself is a group, with function composition as the operation.

(9) The function $f\colon\ \mathbb{Z}/\equiv_{10} \to \mathbb{Z}/\equiv_{10}$ defined by $f([x]) = [2x]$ is not injective or surjective.

(10) The function $f\colon\ \mathbb{Z}/\equiv_{10} \to \mathbb{Z}/\equiv_{10}$ defined by $f([x]) = [3x]$ is injective and surjective.

(11) There is a bijection from the interval $(-10,10)$ to the interval $(1,2)$.

(12) There is a bijection from the interval $[0,1]$ to the interval $(0,1)$.

(13) Let X be a set. There is a bijection from $\mathscr{P}(X)$ to the set $\mathscr{F}$ of all functions $X \to \{0,1\}$.

(14) Let G be the group of all permutations of the set $\{1,2,3,4\}$. Let $g_1 = (1\,2)(3\,4)$ and $g_2 = (2\,3)$. Let H be the set of all permutations in G which can be written in the form of an alternating composition of g_1 and g_2, beginning with g_1. That is:

$$H = \{\mathrm{id}, g_1, (g_2 \circ g_1)^k, g_1 \circ (g_2 \circ g_1)^k : k \in \mathbb{N}\}.$$

Prove that H is a subgroup of G.

(15) Every element of $\mathbb{N}^*$ is either even or is one more than an even number.

(16) Every element of $\mathbb{N}^*$ is either a multiple of 3, one more than a multiple of 3, or two more than a multiple of 3.

(17) There are $n!$ permutations of an n-element set.

(18) Every permutation of the set $\{1,\ldots,n\}$ can be written as the composition of transpositions.

(19) A convex polygon having $n \geq 3$ sides can be triangulated with $n-2$ triangles.

(20) (CHALLENGE) Suppose that G is a group with operation $\circ$ and that H is a subgroup. For $a,b \in G$ define $a \sim b$ if and only if $a \circ b^{-1} \in H$. Prove that $\sim$ is an equivalence relation on G.

(21) (CHALLENGE) Let G be a graph with vertices V and edges E. A cycle in G is a path $v_0,\ldots,v_n$ such that $v_0 = v_n$ but no other vertex is repeated. Suppose that there exist paths $v_0,\ldots,v_n$ and $w_0,\ldots,w_k$ from a vertex a to a vertex b such that if $v_i = w_j$ then $i = j = 0$ or $i = n$ and $j = k$. Prove that G contains a cycle.

(22) (CHALLENGE) Let X be the set of all real-valued functions on the vertices V of a graph G having directed edges and let Y be the set of all real valued functions on the edges E of G. If e is a directed edge of G, let e_- be the tail of e and e_+ be the head of e. Define $\nabla\colon\ X \to Y$ by declaring $\nabla(f)\colon\ E \to \mathbb{R}$ to be the function defined by $\nabla(f)(e) = f(e_+) - f(e_-)$ for all $e \in E$. Prove that ∇ is surjective if and only if G has no cycles.