MA 274

Fall 2016 Exam 1

You may not use textbooks, notes, or refer to other people (except the instructor). You may not have any internet-connected or connectable devices visible to you or others. If you find a problem statement ambiguous, feel free to ask for clarification. Remember to make the structures of your proofs clear.

Please do not place any answers on the exam itself. Instead turn in your final answers in the proper order, and with all preliminary work clearly labelled and attached at the very end of what you turn in. Turn in this cover sheet as the first page of your exam.

Place your name and problem number on every page. You do not have to recopy the problem statement.

Problem	Score	Possible
1		15
2		15
3		10
4		20
5		15
6		15
7		15
8		15
9		10(extra-credit)
Total		120

Name (printed): ____

DEFINITIONS

A **graph** is a set $G = (V, E)$ where V and E are sets and the elements of V are called **vertices** and the elements of E are called **edges**. Additionally, every edge $e \in E$ has two endpoints, each of which is a vertex in V.

An integer a is a **multiple** of an integer k (and k is a **divisor** of a) if there exists $m \in \mathbb{N}$ such that

$$a = mk.$$

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(1) (15 pts) Give a precise definition of the following terms:

(a) **Converse** of a statement $P \Rightarrow Q$.
(b) **Union** of sets A_λ for $\lambda \in \Lambda$ (where Λ is a set).
(c) **Function** $f\colon X \to Y$

(2) (15 pts) Write the negations of the following statements. Phrase your answer as positively as possible.

(a) There exist two lines L_1 and L_2 such that L_1 and L_2 have the same slope and $L_1 \cap L_2 = \varnothing$.
(b) For every $r \in \mathbb{R}$ with $r > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.
(c) If $|x - 5| < .001$, then $|\ln(x) - \ln(5)| < .01$.

(3) (10 pts) Suppose that A_λ is a set for all $\lambda \in \Lambda$ (where Λ is some index set.) Prove that

$$\bigcup_{\lambda \in \Lambda} (A_\lambda^C) = \left(\bigcap_{\lambda \in \Lambda} A_\lambda \right)^C.$$

(4) (20 pts)You may assume the following (in addition to basic algebra):

- if n is a natural number, then precisely one of the following holds: n is a multiple of 3, n is one more than a multiple of 3, or n is two more than a multiple of 3.
- if $r \in \mathbb{Q}$, then there exists $a, b \in \mathbb{Z}$ such that $b \geq 1$, $r = a/b$, and the only common divisors of a and b are -1 and $+1$.

. Prove both of the following:

(a) A natural number $n \in \mathbb{N}$ is a multiple of 3 if and only if n^2 is a multiple of 3.
(b) The number $\sqrt{3}$ is irrational. (i.e. $\sqrt{3} \notin \mathbb{Q}$)

(5) (15 pts) Suppose that X and Y are non-empty sets such that $X \times Y = Y \times X$. Prove that $X = Y$.

(6) (15 pts) Suppose that G is a graph (see the definition on the first page of the exam.) Recall that if a and b are vertices of G, then a **path** from a to b is a list of vertices

$$v_0, v_1, v_2, \ldots, v_n$$

such that:

- $v_0 = a$
- $v_n = b$
- for every $i \in \{0, \ldots, n-1\}$, the vertices v_i and v_{i+1} are endpoints of an edge in G.

Prove that if a, b, and c are vertices in G such that there is a path from a to b and a path from b to c, then there is also a path from a to c.

(7) (15 pts) Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are injective functions. Prove that the function

$$g \circ f\colon X \to Z$$

is injective.

(Exam continues on reverse)

(8) (15 pts) Suppose that X is a set. We say that X is **ultra-nested** if it has the following properties:

(N1) every element of X is a set
(N2) for every $A \in X$, the set $A \cup \{A\} \in X$.

Do the following:

(a) (5 pts) Suppose that $A = \{1,2,3,4,5\}$. Write out the elements of $A \cup \{A\}$.
(b) (10 pts) Prove that if Ξ is a set[1] such that each element $X \in \Xi$ is ultra-nested, then the set

$$\bigcap_{X \in \Xi} X$$

is ultra-nested.

(9) (10 pts Extra-Credit) Suppose that $f\colon X \to Y$ is a bijective function and that $g\colon Y \to X$ and $h\colon Y \to X$ are bijective functions such that the following hold:

- For all $x \in X$, $g \circ f(x) = x$
- For all $x \in X$, $h \circ f(x) = x$.

Prove that for all $y \in Y$, $g(y) = h(y)$. **Also explain** why this implies that a bijection has a *unique* inverse function.

[1]The Greek Letter Ξ is called "xi" and pronounced "ex-see".