F18 MA 274: Exam 3 Study Questions

These questions mostly pertain to material covered since Exam 2, though I have also included some from prior to Exam 1. Some of these are new. The final exam is cumulative, so you should also study earlier material. Some solutions will be posted at a later date - so get started now!

(1) Know the definitions on the website. Any other definitions that you need will be given to you.

(2) When you write a proof, focus on getting the organization clear and correct. If you have to skip some steps or make an assumption that you don't know how to prove, clearly state that that is what you are doing.

(3) Know the theorems we've proved in class and the more significant theorems from the homework.

(4) Don't try to memorize proofs. Instead remember the structure of the proof (proof by contradiction, proof of uniqueness, element argument, etc.) and two or three key steps of the proof. Then at the exam recreate the proof.

(5) At the exam, leave time to write up a nicely written version of each proof. You should have enough time to sketch your ideas out on scratch paper before writing a final version of the proof.

(6) Study the previous study guides and exams as well as your homework, class notes, and the sections of the text we covered.

Prove the following:

(1) The number $\sqrt{2}$ is irrational.

(2) There are infinitely many prime numbers.

(3) There is no set U such that $A \in U$ if and only if A is a set. (Russell's Paradox)

(4) The Halting Problem

(5) DeMorgan's Laws

(6) Suppose G is a group with operation $\circ$ and that $a \in G$. If $f, g \in G$ have the properties that $f \circ a = a \circ f = a$ and $g \circ a = a \circ g = a$, then $f = g$. (That is, the identity in a group is unique.)

(7) Suppose that G is a graph and that a, b, and c are vertices. Then if there is a path from a to b and a path from b to c, then there is a path from a to c.

(8) The intersection of subgroups is a subgroup

(9) The intersection of convex sets is convex

(10) The intersection of event spaces is an event space.

(11) $X \times Y = Y \times X$ if and only if either $X = Y$ or one of X or Y is empty.

(12) If (x_n) is a sequence in a set X such that $\mathrm{range}(x_n)$ is infinite, then (x_n) has a subsequence (x_{n_k}) which is injective and such that $\mathrm{range}(x_{n_k}) = \mathrm{range}(x_n)$.

(13) Suppose that (x_n) is a sequence in $\mathbb{R}$ with the property that for all $N \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x_m < \min\{x_1, \ldots, x_N\}$. Prove that (x_n) has a subsequence (x_{n_k}) which is strictly decreasing.

(14) Prove that if G is a finite, connected, non-empty planar graph, then the number of vertices minus the number of edges plus the number of faces equals 2.

(15) Prove that for every natural number $n \geq 2$, there exist prime numbers $p_1, p_2, \ldots, p_k$ such that $n = p_1 p_2 \cdots p_k$.

(16) Prove that for every rational number $r \in \mathbb{Q}$, there exist $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $r = a/b$ and a and b have no common factor.

(17) Prove that if a and b are natural numbers, then there exist $q, r \in \mathbb{N}^*$ such that $b = aq + r$ and $r < a$.

(18) Prove that if e is an edge of a connected graph G and if $G - e$ results from removing the edge e (but leaving its endpoints) then $G - e$ has at most two components.

(19) Prove that if α is a path from a vertex a to a different vertex b in a graph G, then either α does not pass through any vertex twice or there is a path from a to b which contains fewer vertices than α.

(20) Prove that a connected graph where every vertex has even degree has an Euler circuit.

(21) If X is a subset such that there is an injection $f\colon\ X \to B$ where B is a proper subset of X, then X is infinite.

(22) Recall that the cardinality $|X|$ of a non-empty finite set X is a number n such that there exists a bijection $\{1, \ldots, n\}$. Prove that the number n is unique and prove that if $|X| = |Y|$ then there exists a bijection from X to Y.

(23) Prove that if n is a natural number, then there exists $m \in \mathbb{N}$ and digits $d_i \in \{0, 1, 2\}$ for $i \in \{0, \ldots, m\}$ such that

$$n = \sum_{i=0}^{m} d_i 3^i$$

(In other words, natural numbers can be written in ternary notation.)

(24) Prove that a subset of a countable set is countable.

(25) Prove that the following sets are countable:

(a) $\mathbb{Z}$
(b) $\mathbb{N} \times \mathbb{N}$
(c) $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$
(d) $\bigcup_{\lambda \in \Lambda} A_\lambda$ where Λ is a non-empty countable set and each A_λ is non-empty and countable.
(e) $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k}$

(26) Prove that for every set X, $\operatorname{card} X < \operatorname{card} \mathcal{P}(X)$.

(27) The set of sequences in $\{0, 1\}$ is uncountable.

(28) The interval $[0, 1)$ is uncountable.

(29) If X is an infinite set, then there exists an infinite injective sequence in X

(30) If X has an infinite injective sequence in X, then for any element $a \in X$, $\operatorname{card} X = \operatorname{card} X \setminus \{a\}$.

(31) If X and Y are sets such that there is an injection $f\colon X \to Y$, then there exists a surjection $g\colon Y \to X$.

(32) Let S^1 be the unit circle. For any $\alpha \in \mathbb{R}$, let R_α be the counterclockwise rotation by α radians. (If $\alpha < 0$ this means rotate by $|\alpha|$ radians clockwise.) Suppose that $\theta \in \mathbb{R}$. Let (x_n) be the sequence in S^1 where $x_0 = (1, 0)$ and $x_n = R_\theta(x_{n-1})$ for all $n \in \mathbb{N}$. Prove the following:

   (a) The sequence (x_n) is injective if and only if $\theta \notin \pi\mathbb{Q}$ (i.e. θ is not a rational multiple of π.)
   (b) The sequence (x_n) is periodic (i.e. there exists $n \in \mathbb{N}$ such that $x_n = x_0$) if and only if θ is a rational multiple of π.
   (c) The sequence (x_n) is not surjective.
   (d) ~~If $\theta \notin \pi\mathbb{Q}$, then there exists a subsequence (x_{n_k}) converging to x_0.~~

(33) Prove that the set of algebraic numbers is countable and, therefore, that the set of transcendental numbers is uncountable.

(34) Let $X = \mathcal{P}(\mathbb{R})$ and define $\sim$ on X by $A \sim B$ if and only if there exists a bijection $f\colon A \to B$. Prove that $\sim$ is an equivalence relation.

(35) Let X be a non-empty set and let $\mathcal{F}$ be the set of bijections of X to itself (i.e. permutations of X). For $f, g \in \mathcal{F}$ define $f \sim g$ if and only if there exists a bijection $h \in \mathcal{F}$ such that

$$f = h^{-1} \circ g \circ h.$$

Prove that $\sim$ is an equivalence relation.

(36) State and prove LaGrange's theorem.