Fall 2017/MA 122 WHW 7
Name:______________________________
Time spent:______________________________
People I worked with: ______________________________

None of your answers should be on this sheet. At the top of your solutions, write your full name and the total time spent on this assignment. All answers must be very legibly written, in the correct order, and with the problem number clearly marked. Also list the names of people you worked with on this assignment, as a way of giving credit to them for their contributions.

I suggest that each day you look over the problems and figure out which ones seem connected to what we've been doing in class. Try those first!

If you spend more than 6 hours working on this assignment and do not complete it, you should stop working and email your professor. Remember to write the total amount of time this week you spent working on homework for this course (including WebAssign homework) at the top of your solutions.

The problems are divided into "Scales", "Etudes", and "Concerti". Scales are similar to problems assigned in Webwork: they provide practice in basic computational skills. Etudes require more thought, discussion, and writing. Concerti are very hard problems, worth a very small part of the grade. You should plan on trying them, explaining how far you got towards a solution, and where you got stuck (if indeed you did get stuck.) Getting every Scale and Etude correct constitutes mastery of the course material thus far. In particular, not receiving full credit on "concerti" does not mean that you don't understand the material.

Read Sections 15.1, 16.1, and 16.2 of the text!

I. SCALES (30%)

(1) Let $f(x,y) = e^{-(2x^2+y^2)}$. Find and classify the critical points of f.

(2) Find the points on the surface $z^2 = xy + 1$ which are closest to the origin.

(Hint: Define $s(x,y) = x^2 + y^2 + (xy+1)$. Explain why s is the square of the distance from a point on the surface to the origin. Find the local minima of s and explain why those points are the points on the surface closest to the origin.)

(3) Do problems 5 - 9 from Section 16.1.

(4) Do problems 1 - 6, and 11 - 12 from Section 16.2.

(5) Do problems 13 - 20 from Section 16.2.

(6) Let R be the rectangle $[1,3] \times [-1,1]$. Compute $\iint_R x(y^2+1)\,dA$.

(7) Let T be the triangle bounded by the lines $y = x$, $x = 0$, and $x = 3$. Compute $\iint_T x(y^2+1)\,dA$

II. ETUDES (67%)

(1) (In honor of my trip to West Point) Do problem 14 from Section 15.1.

(2) Do problems 16, 18, 31 from Section 15.2.

(3) Consider the lines:

$$L(s) = \begin{pmatrix} 2s \\ 1s \end{pmatrix}$$

and

$$M(t) = \begin{pmatrix} 3t+1 \\ -2t-3 \end{pmatrix}$$

(so each real number s gives a point on the line $L(s)$ and each real number t gives a point on the line $M(t)$.)

Find the s and t value so that the points $L(s)$ and $M(t)$ are as close together as possible.

(4) Let $f(x,y) = \sin(xy)$. Let U be the disc of radius 1 centered at the origin in $\mathbb{R}^2$. Without doing any calculations, explain why $\iint\limits_U f\,dA = 0$.

(5) Write an explanation of the difference between double integrals and iterated integrals and explain the relationship between them. You explanation should be only 1 - 2 paragraphs and should be addressed to someone who took Calculus long ago and only barely remembers what an integral is.

(6) Let $f(x,y) = xe^y$. Let V be the region in $\mathbb{R}^2$ bounded by the curves $y = x^2$, $y = -x^2$, and $x = 4$. Compute $\iint\limits_V f\,dA$.

(7) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous positive function. Let R be the area under the graph of f for $a \leq x \leq b$. In Calc 1, you learned that the area of R is given by $\int\limits_a^b f(x)\,dx$. In our class, we recently learned that the area of R is also given by $\iint\limits_R 1\,dA$. Use iterated integrals to explain why these two expressions give the same answer.

(8) Let $R = [0,1] \times [0,1]$ in $\mathbb{R}^2$ and suppose that f is some continuous function defined on R. Recall that we defined a Riemann sum of f over R to be as follows:

$$\iint\limits_R f\,dA = \lim_{(\Delta x, \Delta y) \to (0,0)} \sum_{i,j} f(x^*_{i,j}) \Delta x \Delta y.$$

We subdivided R into small squares $R_{i,j}$, each with dimensions $\Delta x \times \Delta y$. From each square $R_{i,j}$ we chose a sample point $x^*_{i,j}$. In this problem, you'll explore why it is important that we have both Δx and Δy converge to 0 and not just the area.

Let $f(x,y) = y$ for every $(x,y) \in R$.

(a) Compute $\iint\limits_R f\,dA$.

(b) For each natural number n, subdivide the interval $[0,1]$ into n points:

$$x_0 = 0, x_1 = 1/n, x_2 = 2/n, x_3 = 3/n, \ldots, x_n = 1.$$

Subdivide R into rectangles, each of height 1 and with bases the intervala $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, etc. Draw a picture of this. Call these rectangles R_1, R_2, etc.

(c) In each rectangle R_i, let $(x_i^*, y_i^*) = (x_i, 0)$. Let $\Delta x = \frac{1}{n}$ and $\Delta y = 1$ and $\Delta A = \Delta x \Delta y$. Compute

$$S(n) = \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A.$$

(d) Show that $\Delta A \to 0$ as $n \to \infty$ and also find

$$\lim_{\Delta A \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A.$$

(e) Explain why the preceding is not equal to $\iint\limits_R f\,dA$, even though we did almost everything right in terms of setting up a Riemann sum.

III. Concerti (3%)

(1) In many real-life optimization problems, there are constraints on the solutions that go beyond the function being optimized. The method of "LaGrange multipliers" is a useful technique for solving many of these types of problems. Read Section 15.3 and do the following:

- Find the maximimum and minimum values of $f(x,y) = 4x + 6y$ given that $x^2 + y^2 = 13$.
- Find the maximum and minimum values of $f(x,y) = x^2 y$ given that $x^2 + 2y^2 = 6$.

(2) Let R be a rectangle in the plane and let f be continuous function defined on R. The average value of f on R is defined to be:

$$\frac{1}{\text{Area}(R)} \int_A f\,dA.$$

In this problem you will explore why this makes sense:

(a) Suppose that R has been divided into n subrectangles each of area $\Delta A = \text{Area}(R)/n$. In each rectangle R_i, choose a point $\mathbf{x}_i$. Write an expression for the average of the values of $f(\mathbf{x}_i)$.

(b) Show that the limit of your answer to (a) as $n \to \infty$ is equal to

$$\frac{1}{\text{Area}(R)} \int_A f\,dA.$$