Fall 2017/MA 397 HW 3: Counting and Directed Graphs

Remember that, although you are encouraged to work together, all of your write-ups must be your own (no copying someone else's solution - not even with minor wording changes.) **List the names of everyone you worked with on the HW!** You are encouraged to **not** look online for solutions - your time is better spent wrestling with the proof yourself or getting help from the professor, than squandering it online.

1. Reading

- Read Sections 1.2 and 1.3 (pages 34 - 66) of the text. You may skip unassigned problems. For now you may also skip the section on "graphic sequences".
- Memorize the definition of **degree** of a vertex, k**-regular graph**, **neighborhood of a vertex**, **order of a graph**, **size of a graph**, **Degree-Sum Formula**, **optimization problem in graph theory**, H**-free**, **directed graph**, **finite automaton**, **Markov chain**, **strongly connected**, **kernel of a digraph**, **oriented graph**, **tournament**.
- Study the proofs of Propositions 1.3.3, 1.3.9, 1.3.15, 1.3.19, 1.4.16, 1.4.29. We'll go over some of these in class and see below for some exercises associated with some of them.

2. To do

Reading Comprehension (your answers need not be long):

(1) Read Example 1.3.16. Explain why the graph $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$ gives an example (for each n) of a simple n vertex graph having $\delta(G) < (n-1)/2$ which is not connected. Explain why this means Proposition 1.3.15 is sharp.

(2) Read Example 1.3.20 and explain what is meant by "getting stuck in a local maximum."

(3) Read Theorem 1.3.23.

   (a) Explain how the degree sum formula is used to obtain the inequality:
$$\sum_{v \notin N(x)} d(v) \geq e(G).$$

   (b) Explain why $e(G) \leq (n-k)k$ (where $k = \Delta(G)$ is the degree of x and is also the maximum degree of any vertex in x.)

   (c) The last part of the proof involves moving a vertex from one partite set in a complete bipartite graph to the other set. Explain why this is useful in the proof.

(4) Read Examples 1.3.24 and 1.3.26. Explain what "the induction trap" is in a failed proof by induction. Have you ever seen such a mistake in an induction proof before?

(5) Read Applications 1.4.4, 1.4.5, and 1.4.14. Pick an example of a game, economic system, or internet-style network and explain how some feature(s) of it can be modelled using a Markov chain. If u and v are vertices (i.e. nodes) of the Markov chain, interpret a uv-path in terms of the original game/system/network.

Problems:

(1) 1.3.3

(2) 1.3.9

(3) 1.3.18

Hint: Fill in the details in the following sketch (or find your own proof.) Assume that the k-regular bipartite graph G is connected. Let (X,Y) be a bipartition of G. Suppose that e is a cut edge with H_1 and H_2 the components of $G-\{e\}$. Notice that $(X\cap V(H_i), Y\cap V(H_i))$ is a bipartition of H_i for $i=1,2$. Without loss of generality, assume that e has an endpoint in $X\cap V(H_1)$. Argue that the number of edges in H_1 is equal to $k|X\cap V(H_1)|-1$. Thus, $|E(H_1)|$ (which is equal to that number) is not a multiple of k (assuming $k\geq 2$)

Now argue that $|E(H_1)| = k|Y\cap V(H_1)|$. But this means that the number of edges in H_1 is a multiple of k, a contradiction.

(4) 1.4.4

(5) 1.4.5

(6) 1.4.8

(7) 1.4.21

(8) 1.4.27