## **Fall 2017/MA 397 Take-home Exam 2.**

This exam is distributed on Wednesday, Nov. 8 and is due on Wednesday, Nov. 17 by 5 PM.

**The rules:** You may use your textbook and class notes. You may not use the internet, including (but not limited to) sites where math questions can be posted. You also may not use the solutions manual to the textbook or books from the library. If you are stuck on a problem, you may ask for a hint. The cost of the hint is 5% of the total number of points for the problem. (So if you write a perfect solution after receiving the hint, your grade would be a 95% on that problem.) You may ask for multiple hints on a given problem, each hint costs the same number of points. Questions of clarification are free. You may not discuss this exam with anyone except the instructor.

**Added 11/10/17:** You may also use Otto Bretscher's Linear Algebra text if you need to review kernels and images of linear maps. You may also use any notes you took while taking linear algebra at Colby or another institution. You may not search the internet for linear algebra notes, facts, etc.

Your solutions should be submitted in order, one problem per page, with your name on every page. You must sign this cover sheet **before** taking the exam and include it with your exam solutions.

I have read and will abide by the rules for the exam.

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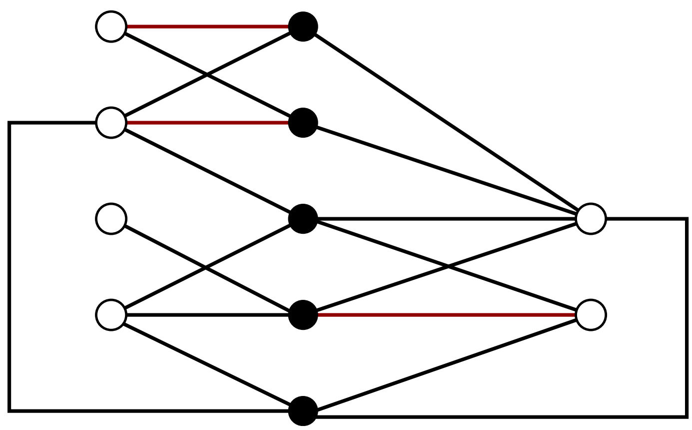
Signature

## 1. CALCULATIONS AND ALGORITHMS

You need to do 3 out of the following 4 problems.

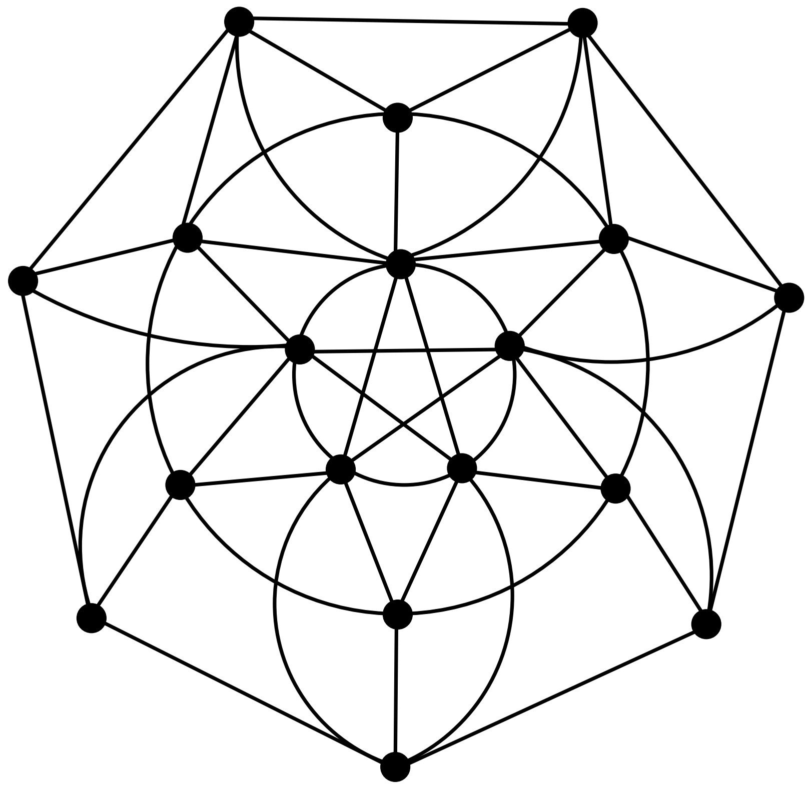
(1) The $X,Y$-bipartite graph below has a matching $M$ with 3 edges (indicated in red). The sets $X$ and $Y$ are indicated with different markers for vertices. Apply the Augmenting Path algorithm (Algorithm 3.2.1) to the graph twice, each time finding an $M$-augmenting path and then using that path to find a new matching. For each step of the algorithm (both times you apply it) keep track of the sets of $S \subset X$ and $T \subset Y$ of vertices which have been reached. For each reached vertex also record the vertex from which it is reached.



(2) The complete bipartite graph $K_{6,6}$ with bipartition $X,Y$ has edges weighted according to the matrix below. Apply three iterations of the Hungarian Algorithm (Algorithm 3.2.9). For each iteration, draw the equality subgraph, the maximum matching in the equality subgraph, and the associated vertex cover. Also keep track of the tolerance $\varepsilon$.

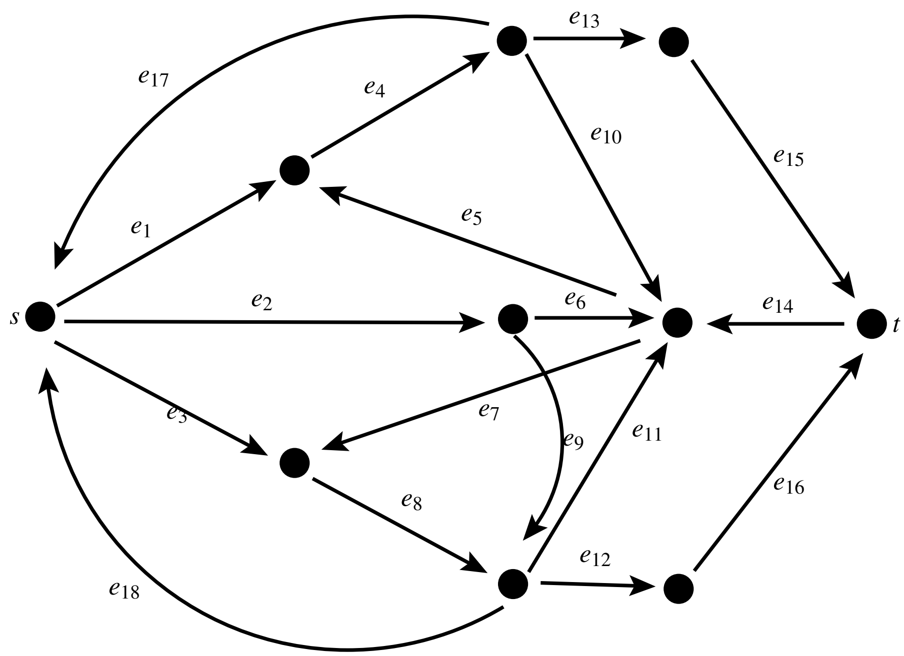
$$\begin{pmatrix} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ \hline x_1| & 1 & 5 & 6 & 0 & 1 & 4 \\ x_2| & 0 & 0 & 4 & 1 & 1 & 1 \\ x_3| & 5 & 4 & 5 & 3 & 4 & 0 \\ x_4| & 2 & 1 & 0 & 0 & 0 & 5 \\ x_5| & 3 & 3 & 1 & 1 & 1 & 1 \\ x_6| & 5 & 5 & 0 & 0 & 5 & 3 \end{pmatrix}$$

(3) Determine the connectivity of the graph below. Be sure to give a complete justification for your answer.



(4) A network is drawn below with source vertex $s$ and sink vertex $t$. Each edge of the network is labelled. Below the figure is a list of capacities $c(e)$ and flow values $f(e)$ for each edge of the network. Apply the Ford-Fulkerson algorithm (Algorithm 4.3.9) as many times as necessary to to find a maximal flow. For each output of the algorithm, draw the resulting $f$-augmenting path, compute its tolerance, and either list the flow values for the edges for the new flow or highlight the vertices in the cut certifying that you've found a maximal flow.

Note: It may be possible for you to simply inspect the network to find a maximal flow. This problem is explicitly testing your ability to follow the Ford-Fulkerson algorithm.




| edge | $c(e)$ | $f(e)$ |
|---|---|---|
| $e_1$ | 2 | 1 |
| $e_2$ | 5 | 4 |
| $e_3$ | 3 | 3 |
| $e_4$ | 6 | 2 |
| $e_5$ | 5 | 1 |
| $e_6$ | 6 | 3 |
| $e_7$ | 8 | 3 |
| $e_8$ | 8 | 6 |
| $e_9$ | 3 | 1 |
| $e_{10}$ | 1 | 0 |
| $e_{11}$ | 5 | 0 |
| $e_{12}$ | 5 | 0 |
| $e_{13}$ | 4 | 2 |
| $e_{14}$ | 3 | 1 |
| $e_{15}$ | 5 | 2 |
| $e_{16}$ | 5 | 0 |
| $e_{17}$ | 1 | 0 |
| $e_{18}$ | 7 | 7 |

## 2. PROOFS

Prove all 3 of the following.

(1) Let $G$ be a 2-connected simple graph and let $xy \in E(G)$ be an edge with endpoints $x$ and $y$. **Prove** that $G - xy$ is 2-connected if and only if $x$ and $y$ lie on a cycle in $G - xy$. Also **prove** (as a consequence of the previous result) the following: Suppose $G$ is 2-connected. Then it has the property that for every edge $e \in E(G)$, the graph $G - e$ is not 2-connected if and only if it has the property that every cycle is an induced subgraph. (That is: iff there is no "chord" for any cycle.)

(2) Show that the $M$-augmenting path algorithm (Algorithm 3.2.1) is a special case of the Ford-Fulkerson Algorithm (Algorithm 4.3.9). To do this, as a first step, you need to specify how you are transforming the problem of finding an $M$-augmenting path in a bipartite graph with a matching $M$ into a network flow problem. You then need to show how each step of the path algorithm corresponds to a step of the Ford-Fulkerson algorithm.

(3) The point of this problem is to give you some practice working with the cohomology vector spaces of a graph. Prove the following:

(a) Let $V$ be a finite dimensional vector space (with real coefficients) and let $W \subset V$ be a subspace. Let $\iota \colon W \to V$ be the inclusion map. (That is $\iota(w) = w$ for all $w \in W$.) Let $\phi \colon V \to V/W$ be the map taking an vector $v \in V$ to its equivalence class in the quotient vector space $V/W$. Consider:

$$0 \to W \xrightarrow{\iota} V \xrightarrow{\phi} V/W \to 0$$

Use the fact that both $\iota$ and $\phi$ are linear maps and the rank-nullity theorem to prove that

$$\dim V/W = \dim V - \dim W.$$

As a reminder:

**Theorem** (Rank-Nullity). *Suppose that $f \colon A \to B$ is a linear map between finite dimensional vector spaces. Then*

$$\dim \ker f + \dim \operatorname{im} f = \dim A.$$

As a reminder: $\ker f = \{a \in A : f(a) = 0\}$ and $\operatorname{im} f = \{b \in B : \exists a \in A, f(a) = b\}$. These are both subspaces of $A$ and $B$ respectively.

(b) Recall that for a digraph $G$, the vector space $C^0(G)$ consists of all real-valued functions on the vertices of $G$. Also $C^1(G)$ consists of all real valued functions on the edges of $G$. Given $f \in C^0(G)$, we define $\nabla f \in C^1(G)$ by letting $F(e) = f(v_+) - f(v_-)$ where $v_+$ is the head of $e$ and $v_-$ is the tail of $e$. Notice that $\nabla \colon C^0(G) \to C^1(G)$ is a linear map. If $\alpha$ is a path in $G$ and $F \in C^1(G)$, define

$$\int_\alpha F = \sum_{e \in \alpha} \varepsilon(e) F(e)$$

where the sum is over all edges $e$ in $\alpha$ and $\varepsilon(e) = +1$ if $\alpha$ travels in the same direction as $e$ and $\varepsilon(e) = -1$ if $\alpha$ travels in the opposite direction across $e$.

Suppose that $\alpha$ is a path from a vertex $v$ to a vertex $w$ in $G$ and that $f \in C^0(G)$. **Prove** that $\int_\alpha \nabla f = f(w) - f(v)$. **Explain** why this is like the fundamental theorem of calculus and **conclude** that if $\alpha$ is a closed loop then $\int_\alpha \nabla f = 0$.

As a reminder:

**Theorem** (FTC 2)**.** *If* $h\colon \mathbb{R} \to \mathbb{R}$ *and* $H$ *is any anti derivative of* $h$*, then* $\int_a^b h\,dx = H(b) - H(a)$.

(c) Suppose that $G$ is a digraph and that $C$ is a cycle in the underlying multigraph of $G$. **Prove** that there is $F \in C^1(G)$ such that $\int\limits_C F \neq 0$. **Conclude** that $F$ is not in the image of $\nabla$.

(d) Let $G$ be a digraph whose underlying graph is a tree. Let $F \in C^1(G)$. Prove that there exists $f \in C^0(G)$ such that $F = \nabla f$.

(e) Recall that $H^1(G) = C^1(G)/\operatorname{im}\nabla$. Prove that $\dim H^1(G) = 1$ when $G$ is a cycle. Explain why this shows that $H^1(G)$ is isomorphic to $\mathbb{R}$.

(Hint: To show that $\dim H^1(G) = 1$ you need to show that if $F$ and $F'$ are in $C^1(G)$ and are not in $\operatorname{im}\nabla$, then there exists a non-zero scalar $a$ and an $f \in C^0(G)$ such that $F' = aF + \nabla f$. Use the previous part to make the work simpler.)