Fall 2017/MA 397 Take-home Exam 1.

This exam is distributed on Friday, Sept. 29 and is due on or before October 11.

The rules: You may use your textbook and class notes. You may not use the internet, including (but not limited to) sites where math questions can be posted. You also may not use the solutions manual to the textbook or books from the library. If you are stuck on a problem, you may ask for a hint. The cost of the hint is 5% of the total number of points for the problem. (So if you write a perfect solution after receiving the hint, your grade would be a 95% on that problem.) You may ask for multiple hints on a given problem, each hint costs the same number of points. Questions of clarification are free. You may not discuss this exam with anyone except the instructor.

Your solutions should be submitted in order, one problem per page, with your name on every page. You must sign this cover sheet **before** taking the exam and include it with your exam solutions.

I have read and will abide by the rules for the exam.

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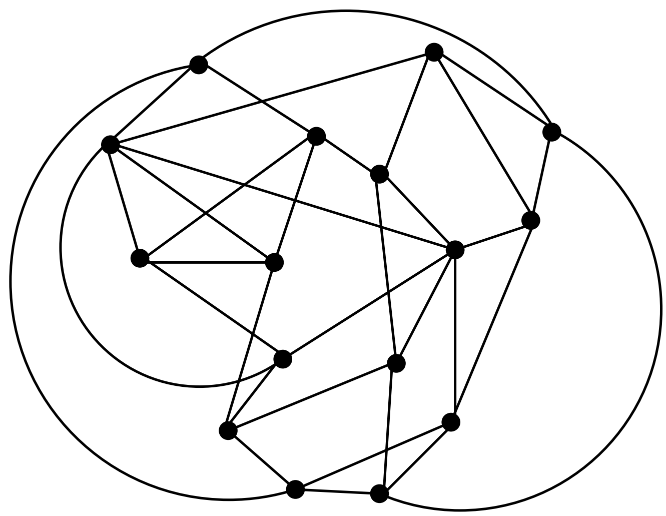
Signature

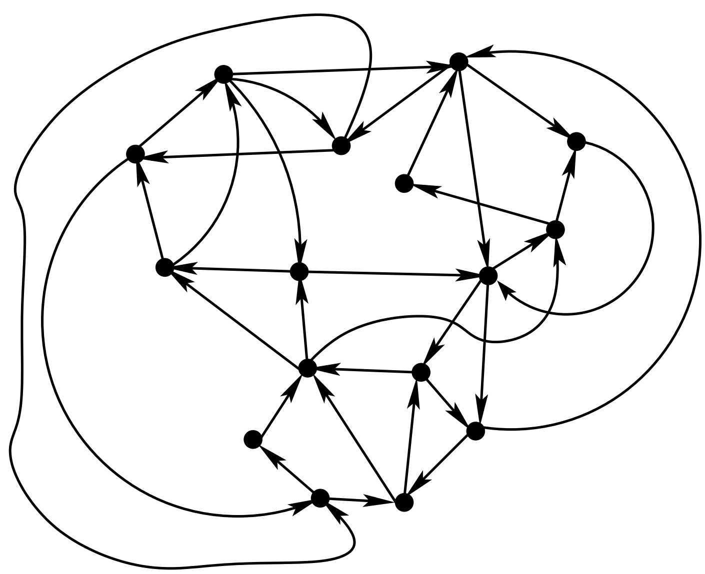
1. Calculations and Algorithms

You need to do 3 out of the following 4 problems.

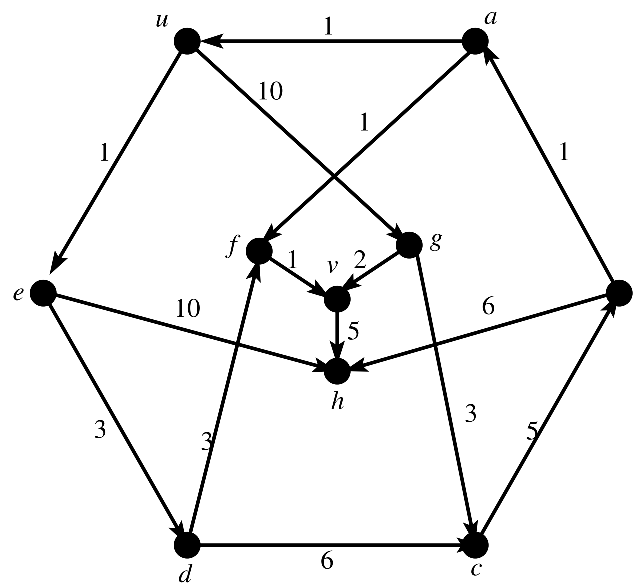
(1) Give a decomposition of the following graph into cycles. You may indicate your answer by drawing the graph using different colors to indicate the different cycles or by labelling the edges and listing the edges in each cycle.



(2) Produce an eulerian trail (i.e. a trail which traverses each edge exactly once) in the following digraph.



(3) In the following weighted version of the Petersen graph, use Dijkstra's algorithm to find weight of a smallest weight path from u to v. For each step of Dijkstra's algorithm, you should record how the "tentative distance" for all the vertices in the graph changes. Once you determine the weight of a smallest weight path from u to v you may stop the algorithm.



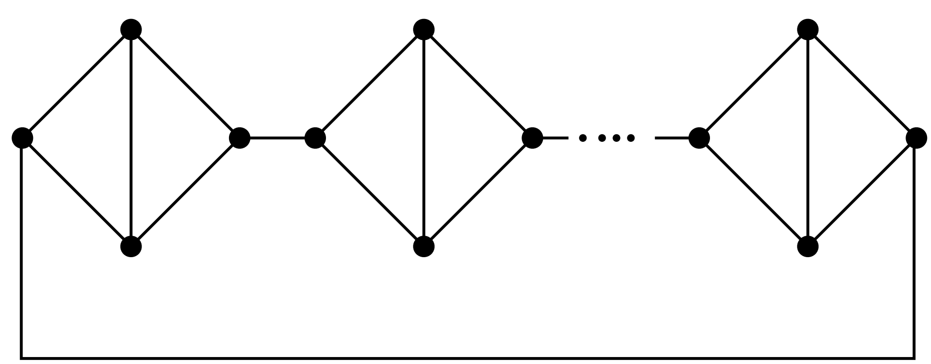
(4) Construct a tree T (with labelled vertices) having Prüfer code $(7,1,2,6,6,6,6,2,7,3)$.

2. Proofs

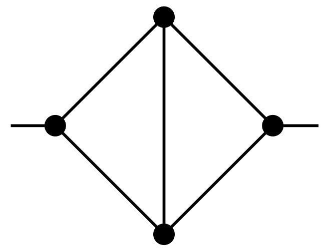
You need to prove 3 out of the following 4 results. A correct proof of the fourth will result in extra-credit.

(1) Suppose that G is a simple graph with $n \geq 2$ vertices. If G has an independent set of size $a \geq 2$, what is the most number of edges G can have (in terms of n and a)? Prove that your answer is correct for all $n, a \geq 2$. Also give an example for $n = 10$ and $a = 6$ to show that your bound is sharp in that case.

(2) In class we considered connected 3-regular graphs having the property that no edge is a cut edge and which were "contraction minimal." (See Problem 1.3.66 and Example 1.3.26) I presented a family of graphs (suggested by the author of the textbook) which satisfied these conditions. They appeared as so:



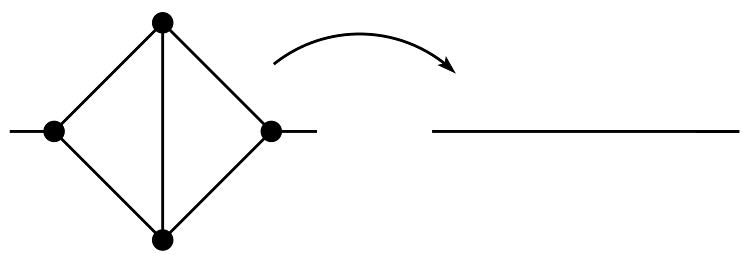
I also presented a different family of graphs which at first I believed to have these properties, but which your classmates pointed out did not. In this question you'll prove that the family of graphs presented by the author of the text is the only family of simple graphs with all of these properties. Recall (from Example 1.1.35 in the book) that a **kite** in a 3-regular graph G is a subgraph as so:



That is, it is a subgraph isomorphic to a kite with the two vertices having degree 2 in the kite joined to other edge(s) of the graph.

(a) Prove the following. Suppose that G is a graph which is 3-regular, connected, has no edge being a cut edge, and which is not obtained by expanding any other graph (as in Example 1.3.26). Prove that G has a kite as an induced subgraph.

(b) Define a "kite removal" to be the graph transformation below. The inverse operation is a "kite insertion."



(c) Explain why if G' is obtained from G by a kite removal then either $G = K_4$ or G' is a simple graph.

(d) Construct an example of a 3-regular graph with no edge a cut edge and which is not obtained by expanding another graph such that G is not in the family of graphs produced by the author of the text. (Hint: Combine expansion with kite insertion.)

(3) Give an algorithm which takes a simple graph with vertices $v_1, \ldots, v_n$ (with $n \geq 2$) and edges $e_1, \ldots, e_m$ and finds a maximal clique. (That is a subgraph which is a complete graph and is not a proper subgraph of any other complete subgraph.) **Also**, give an example of your algorithm in action on the graph obtained from K_6 by removing a single edge.

(4) In class we proved that a connected graph has an euler circuit if and only if every vertex has even degree. Of course, not every graph satisfies this requirement. For each of the following conditions, show that for *any* graph G there is a graph G' such that G is isomorphic to a subgraph of G'; G' has an euler circuit; and

(a) $V(G') = V(G)$ (Note G' may not be a simple graph)
(b) G is an induced subgraph of G' and, out of all such G', $|V(G')| - |V(G)|$ is as small as possible.

Also: Explain how the first construction of G' can be interpreted as saying that in any city the residents could build new roads, without building any new locations, such that they can walk around town, travelling every road exactly once and return home. **And** explain how the second construction of G' means that they can build new destinations in the town and roads leading to the destination, but no new roads between old destinations, such that such a walk is possible.