Fall 2016/MA 314 Metric Spaces: Technical Stuff

The purpose of these notes is to introduce a few technical tools for handling metric spaces. These tools are studied more thoroughly in courses in (point-set) topology and real analysis. These notes are intended to supplement in class discussion.

1. Continuity

As we know from Calculus, continuity is a powerful concept. It turns out that we can define continuous functions for metric spaces as well.

Definition (Continuous function)**.** Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f\colon X \to Y$ is continuous at a point $a \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_X(x,a) < \delta$ then $d_Y(f(x), f(a)) < \varepsilon$. The function f is **continuous** if it is continuous at every $a \in X$. It is a **homeomorphism** if it is a bijection, is continuous, and has continuous inverse.

One way of thinking about this definition is that a continuous function $f\colon X \to Y$ takes points that are near (i.e. within δ) of a to points that are near (i.e. within ε) of a. Thinking of the definition in this way, however, we must be sure to remember that δ is allowed to depend on ε.

We can rephrase this definition, using balls.

Definition (Open ball)**.** An (open) **ball** in a metric space (X,d), centered at $a \in X$, of radius $r > 0$ is the set

$$B_r(a) = \{x' \in X : d_X(x',x) < r\}$$

of elements of X strictly within distance r of x.

A function $f\colon X \to Y$ between metric spaces is then continuous at $a \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(a)) \subset B_\varepsilon(f(a))$. Recall that if $S \subset X$, then $f(S)$ is the subset of Y defined by $f(S) = \{y \in Y : \exists x \in S, y = f(x)\}$.

We present two other versions of continuity.

Definition (topologically continuous)**.** Let (X,d_X) be a metric space. A subset $U \subset X$ is **open** if for every $a \in U$, there exists $r > 0$ such that $B_r(a) \subset U$. A function $f\colon (X,d_X) \to (Y,d_Y)$ is **topologically continuous** if for every open set $U \subset Y$, the set $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X.

We can summarize this definition by saying that the preimage of an open set is open. We can prove that the two definitions are equivalent. Notice that an open ball is an example of an open set (proof?).

Theorem 1.1. *Let (X,d_X) and (Y,d_Y) be metric spaces and let $f\colon X \to Y$ be a function. Then f is continous if and only if it topologically continuous.*

Fill in the blanks for the following proof or write your own.

Proof. Assume that f is continuous. Let $U \subset Y$ be open. We will show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$. We must show that there exists $r > 0$ such that $B_r(x) \subset f^{-1}(U)$.

Since $x \in f^{-1}(U)$, by definition, $f(x) \in U$. Since U is open, there exists _______ such that _______ . Since f is continuous, there exists _______ such that for all $x' \in$ _______, we have $f(x') \in$ _______ $\subset U$. Thus, _______ $\subset f^{-1}(U)$. Hence, f is topological continuous.

Now assume that f is topologically continuous. We will show f is continuous. Let $a \in X$ and let $\varepsilon > 0$. Since _______, the set $f^{-1}(B_\varepsilon(f(a)))$ is open. Since $a \in f^{-1}(B_\varepsilon(f(a)))$, there exists $\delta > 0$ such that _______ . Consquently, if $x \in B_\delta(a)$, we have $f(x) \in B_\varepsilon(f(a))$. Thus, f is continous. □

We can also phrase continuity in terms of sequences.

Definition (sequentially continuous)**.** Suppose that (x_n) is a sequence in X. We say it **converges** to $a \in X$, if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in B_\varepsilon(x)$. A function $f\colon X \to Y$ between metric spaces is **sequentially continuous** if whenever a sequence (x_n) in X converges to an element $a \in X$, the sequence $(f(x_n))$ converges to $f(a)$ in Y.

Theorem 1.2. *A function $f\colon X \to Y$ between metric spaces is continuous if and only if it is sequentially continuous.*

Fill in the blanks or write your own proof.

Proof. Let d_X and d_Y be the metrics on X and Y. Assume first that $f\colon X \to Y$ is continuous. We will show it is sequentially continuous.

Let (x_n) be a sequence in X converging to $a \in X$. We will show $(f(x_n))$ converges to $f(a)$. Let $\varepsilon > 0$. Since _______, there exists $\delta > 0$ such that _______. Since _______, there exists $N \in \mathbb{N}$ such that _______. Consequently, _______. Thus, $(f(x_n))$ converges to $f(a)$.

We will now show that if $f\colon X \to Y$ is sequentially continuous then it is continuous by proving the contrapositive. Assume that f is not continuous. Thus, there exists $a \in X$ and $\varepsilon > 0$ such that for all $\delta > 0$, there is a point $x \in B_\delta(a)$ such that $f(x) \notin B_\varepsilon(f(a))$. In particular, for $\delta = 1/n$ (with $n \in \mathbb{N}$), there exists a point $x_n \in B_{1/n}(a)$ such that $f(x_n) \notin B_\varepsilon(f(a))$.

The sequence (x_n) converges to a because _______.

The sequence $(f(x_n))$ does not converge to $f(a)$ because _______.

Thus, f is not sequentially continuous, as desired. □

Sequential continuity is probably the most natural way of understanding continuity – a sequentially continuous function preserves the convergence of all sequences.

2. Compactness

Let (X,d) be a metric space. Most metric spaces of interest have infinitely many elements, which makes induction or any kind of counting argument difficult. There is a natural class of metric spaces which are not finite but have the property which makes counting arguments possible. The property is called *compactness*. For surfaces (the metric spaces we are most interested in) compactness is equivalent to being made out of finitely many simple-to-understand pieces. Just as there were multiple ways of defining the notion of "continuous function" so there are multiple ways of defining the notion of "compact". We will stick with the definition which, for metric spaces, is most natural. In a point-set topology or real analysis class, this might be called "sequential compactness."

Definition (Compact)**.** Let (X,d) be a metric space. It is **compact** if every sequence in X has a subsequence which converges to some point in X

Informally, this says that the terms of every sequence must pile up somewhere (perhaps in multiple places).

Example 2.1. The sequence $\alpha = 1,0,1,0,1,0,\dots$ in $[0,1] \subset \mathbb{R}$ is a sequence which does not converge. However, it does have a convergent subsequence; for example, $\alpha' = 1,1,1,1,\dots$ On the other hand, the sequence $\gamma = 1,2,3,4,5,\dots$ in $\mathbb{R}$ is a sequence which does not converge and, furthermore, has no convergent subsequence.

The next theorem is crucially important, but we won't prove it.

Theorem 2.2. *The interval* $[0,1]$ *with the euclidean metric is compact.*

Proof. Proof omitted. See a real analysis or point-set topology text. □

Indeed, every closed, bounded interval in $\mathbb{R}$ is compact as follows from the next exercise.

Exercise 2.3. Suppose that X and Y are metric spaces such that X is compact and $f\colon X \to Y$ is surjective and continuous. Then Y is compact.

Compactness is also preserved under products, as indicated in the next theorem.

Theorem 2.4. *Suppose that for* $k \in \{1,\dots,n\}$ *the metric space* (X_k, d_k) *is compact. Then under the metric* d *defined below, the metric space* $X = \times_{k=1}^n X_k$ *is compact.*

For $(x_1,\dots,x_n),(y_1,\dots,y_n) \in X$ define the **euclidean product metric** to be

$$d\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = \sqrt{d_1^2(x_1,y_1) + d_2^2(x_2,y_2) + \cdots + d_n^2(x_n,y_n)}.$$

Lemma 2.5. The euclidean product metric is a metric.

Proof. The properties (M1), (M2), (M3) follow easily. We prove (M4), the triangle inequality. Let $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$, $z = (z_1,\dots,z_n)$ be elements of X. Observe that

$$\Delta(x,y) = \begin{pmatrix} d_1(x_1,y_1) \\ d_2(x_2,y_2) \\ \vdots \\ d_n(x_n,y_n) \end{pmatrix} \in \mathbb{R}^n$$

and that

$$d(x,y) = ||\Delta(x,y)||.$$

Thus,

$$\begin{aligned} d(x,y) + d(y,z) &= ||\Delta(x,y)|| + ||\Delta(y,z)|| \\ &\geq ||\Delta(x,y) + \Delta(y,z)|| \\ &\geq ||\Delta(x,z)|| \\ &= d(x,z). \end{aligned}$$

The last $\geq$ follows from the triangle inequality applied to each of the d_i and the fact that each entry in the the vectors $\Delta(x,y)$, $\Delta(y,z)$, and $\Delta(x,z)$ are non-negative. □

Sketch proof of theorem. We prove this by induction on n. If $n = 1$ the theorem is obvious. Assume $n = 2$. Let $((x_n,y_n))$ be a sequence in $X_1 \times X_2$. Since X_1 is compact, (x_n) has a subsequence (x_{n_k}) converging to some $a \in X_1$. Since X_2 is compact, the sequence (y_{n_k}) has a subsequence $(y_{n_{k_\ell}})$ converging to some $b \in X_2$. Then the subsequence $(x_{n_{k_\ell}}, y_{n_{k_\ell}})$ converges to (a,b).

Assume that the theorem is true for some $n \geq 2$. We'll prove it for $n+1$. Let

$$\phi\colon X_1 \times X_2 \times \cdots \times X_{n+1} \to (X_1 \times X_2 \times \cdots \times X_n) \times X_{n+1}$$

be the function defined by

$$\phi((x_1,\dots,x_{n+1})) = ((x_1,\dots,x_n),x_{n+1}).$$

Give both $X_1 \times X_2 \times \cdots \times X_{n+1}$ and $(X_1 \times X_2 \times \cdots \times X_n) \times X_{n+1}$ the product metrics and observe that ϕ is an isometry. The result follows immediately. □

Corollary 2.6. Any cube in $\mathbb{R}^n$ is compact.

Definition. Suppose that $V \subset \mathbb{R}^n$. We say that V is **closed** if V^c is open.

Theorem 2.7 (Closed sets contain their limits)**.** *The set $V \subset \mathbb{R}^n$ is closed if and only if every sequence in V which converges to $a \in \mathbb{R}^n$ has $a \in V$.*

Proof. Suppose that V is not open. Then no open ball based at a is contained in V^c. Thus, for all $n \in \mathbb{N}$, there exists $v_n \in V$ such that $d(v_n,a) < 1/n$. The sequence (v_n) is a sequence in V converging to $a \in V^c$, so V does not contain all its limit points.

Now suppose that V does not contain all its limit points. Then there is a sequence (v_n) in V converging to some $a \in V^c$. By the definition of convergence, no open ball centered at a is contained in V^c, and so V^c is not open. □

Theorem 2.8. *If X is a compact metric space, and if $V \subset X$ is closed. Then V is compact (with the subspace metric).*

Proof. Let (v_n) be a sequence in V. Then (v_n) is also a sequence in X. By the definition of compact, there exists a subsequence (v_{n_k}) which converges to $a \in X$. Since V is closed, $a \in V$ and so every sequence in V has a subsequence converging to a point in V. Hence, V is compact. □

Corollary 2.9. Suppose that $V \subset \mathbb{R}^n$ is closed and bounded (i.e. there exists $M \geq 0$ such that for all $x,y \in V$, $d(x,y) \leq M$). Then V is compact.

proof sketch. Every bounded set is contained in some cube. Cubes are compact, so V must also be compact. □

Exercise 2.10. Show that if $U \subset \mathbb{R}^n$ is either open or not bounded, then U is not compact.

3. Completeness

Compactness is about guaranteeing that sequences have convergent subsequences. Completeness is about having sequences which "should" converge actually converge.

Definition. Suppose that (x_n) is a sequence in a metric space (X,d). The **discrete length** of (x_n) is

$$L((x_n)) = \sum_{k=1}^{\infty} d(x_k,x_{k+1}).$$

The space (X,d) is **complete** if every finite length sequence converges in X.

Prove the next lemma. You will need to use some properties of series from Calc II.

Lemma 3.1. Suppose that (x_n) is a finite length sequence in a metric space X. If (x_{n_k}) is a subsequence then it is also finite length and if (x_{n_k}) also converges then so does (x_n).

Theorem 3.2. *If (X,d) is compact, then it is complete.*

Proof. Let (x_n) be a finite length sequence. By compactness of X, it has a convergent subsequence. By the previous lemma this means that (x_n) also converges. □

Theorem 3.3. *$\mathbb{R}^k$ is complete for all $k \in \mathbb{N}$.*

Proof. Let (x_n) be a finite length sequence in $\mathbb{R}^k$. Then $S = \{x_n : n \in \mathbb{N}\} \subset \mathbb{R}^k$ must be bounded as (by the polygon inequality):

$$d(x_1, x_n) \leq d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \leq L((x_n))$$

for all $n \in \mathbb{N}$.

Let C be a compact set containing S. Then (x_n) is a sequence in C and therefore has a convergent subsequence. Since (x_n) is finite length that is enough to guarantee that it also converges. Thus, $\mathbb{R}^k$ is complete. □

Exercise 3.4. Suppose that X is any metric space such that for every bounded set $S \subset X$ there exists a compact set $C \subset X$ such that $S \subset C$. Prove X is complete.

4. INFIMA AND SUPREMA

Definition. Suppose that $A \subset \mathbb{R}$. A **least upper bound** for A is an element $r \in \mathbb{R} \cup \{-\infty, \infty\}$ such that, for all $a \in A$, $a \leq r$. An upper bound β for A is the **supremum** of A if it is no larger than any other upper bound for A. We write $\beta = \sup A$. A number $r \in \mathbb{R}$ is a **lower bound** for A if $a \geq r$ for all $a \in A$. A lower bound $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$ for A is the **infimum** of A if it is no smaller than any other lower bound for A. We write $\alpha = \inf A$.

The following is an important property of the real numbers. We omit the proof.

Theorem 4.1. *If $A \subset \mathbb{R}$ has an upper bound $r \in \mathbb{R}$, then $\sup A$ exists (and is a real number). Similarly, if A has a lower bound in $\mathbb{R}$ then $\inf A$ exists and is a real number.*

The next exercise is key to how we use infima and suprema in practice. It shows that decreasing a supremum by a tiny bit allows us to capture an element of the set and increasing an infimum by a tiny bit also allows us to capture an element of the set.

Exercise 4.2. Suppose that $A \subset \mathbb{R}$. If $\sup A \in \mathbb{R}$, then $\beta = \sup A$ if and only if β is an upper bound for A and for all $\varepsilon > 0$ there exists $a \in A \cap (\beta - \varepsilon, \beta)$. Similarly, if $\inf A \in \mathbb{R}$, then $\alpha = \inf A$ if and only if α is a lower bound for A and for all $\varepsilon > 0$ there exists $a \in A \cap (\alpha, \alpha + \varepsilon)$.

We can use infima to define path metrics.

Definition. Suppose that $U \subset \mathbb{R}^2$ is path-connected; i.e. there is a path in U between any two points in U. Let d_{eucl} be the euclidean metric on $\mathbb{R}^2$ and recall that $L(\gamma)$ is the length of a piece-wise differentiable path $\gamma\colon [a,b] \to U$. For $x, y \in U$, define

$$d_{\text{path}}(x,y) = \inf\{L(\gamma) : \gamma \text{ is a piecewise differentiable path from } x \text{ to } y\}.$$

Theorem 4.3. *Let $U \subset \mathbb{R}^2$ be path connected. Then d_{path} is a metric on U.*

In the following proof, the key to proving the triangle inequality is the observation that if $q, r \in \mathbb{R}$ have the property that for all $\varepsilon > 0$ we have $q \leq r + \varepsilon$, then in fact $q \leq r$.

Proof sketch. Let $P(x,y)$ be the set of piecewise differentiable paths in U from x to y. Since U is path-connected, $P(x,y) \neq \varnothing$. Every element of $P(x,y)$ is non-negative since it is the length of a path. Thus, $d_{\text{path}}(x,x) \in [0,\infty)$. The constant path is differentiable and so $d_{\text{path}}(x,x) = 0$ for all $ax \in U$.

$\langle$ *Prove* $d_{\text{path}}(x,y) = 0 \Rightarrow x = y\rangle$

$\langle$ *Establish a bijection between* $P(x,y)$ *and* $P(y,x)$ *which preserves length and use this to show symmetry* $\rangle$

We now prove the triangle inequality. Let $x,y,z \in U$. Let $\varepsilon > 0$. Then there exists a path $\gamma \in P(x,y)$ such that $L(\gamma) \in (d_{\text{path}}(x,y), d_{\text{path}}(x,y) + \varepsilon/2)$. Likewise, there exists a path $\psi \in P(y,z)$ such that $L(\psi) \in (d_{\text{path}}(y,z), d_{\text{path}}(y,z) + \varepsilon/2)$. By re-parameterizing, we may assume that the domain of γ is $[0,1]$ and that of ψ is $[1,2]$. Define

$$\zeta(t) = \begin{cases} \gamma(t) & t \in [0,1] \\ \psi(t) & t \in [1,2] \end{cases}$$

for all $t \in [0,2]$. Then $\zeta\colon [0,2] \to U$ is a path from x to z. Its length is $L(\gamma) + L(\psi)$. Thus, for any $\varepsilon > 0$, there exists $\zeta \in P(x,z)$, such that

$$d_{\text{path}}(x,z) \leq L(\zeta) \leq d_{\text{path}}(x,y) + d_{\text{path}}(y,z) + \varepsilon$$

Since this is true for all $\varepsilon > 0$, we have $d_{\text{path}}(x,z) \leq d_{\text{path}}(x,y) + d_{\text{path}}(y,z)$, as desired. □

4.1. **Hausdorff distance.** We won't use this result, but it's a convenient place to practice using infima.

Suppose that (X,d) is a bounded metric space (i.e. there exists M such that for all $x,y \in X$ $d(x,y) \leq M$.). For non-empty subsets $A,B \subset X$, define the **Hausdorff distance** between A and B to be

$$d_H(A,B) = \inf\{\varepsilon : (\forall b \in B, \exists a \in A \text{ s.t. } d(b,a) < \varepsilon) \text{ and } (\forall a \in A, \exists b \in B \text{ s.t. } d(a,b) < \varepsilon)\}$$

(that is, the infimal ε such that enlarging A by ε contains B and enlarging B by ε contains A.

Theorem 4.4. *Hausdorff distance d_H is a semi-metric on the set of non-empty subsets of a bounded metric space.*

4.2. **Quotient semi-metrics.** This material is crucial.

Let (X,d) be a metric space and suppose that $\sim$ is an equivalence relation on X. For $x \in X$, we let $\overline{x} = \{y \in X : x \sim y\}$ denote the equivalence class of x and $\overline{X} = \{\overline{x} : x \in X\}$ be the quotient set. For $\overline{x}, \overline{y} \in \overline{X}$, let

$$d(\overline{x},\overline{y}) = \inf\{d(x',y') : x' \in \overline{x}, y' \in \overline{y}\}.$$

Example 4.5. Let $X = \mathbb{R}$. Define an equivalence relation $\sim$ on X by

$$x \sim y \Leftrightarrow \begin{cases} x = y & \text{or} \\ \exists n,m \in \mathbb{N} \text{ s.t. } x = 1/n \text{ and } y = 1/m & . \end{cases}$$

Observe that $d(\overline{0},\overline{2}) = d(0,2) = 2$ but

$$d(\overline{2},\overline{1}) + d(\overline{1},\overline{0}) = d(2,1) + \inf\{1/n : n \in \mathbb{N}\} = 1 + 0 = 1.$$

Thus, d, when applied to $\overline{X}$, does not satisfy the triangle inequality.

We want to turn d into something more like a metric on $\overline{X}$. Here's how we do it.

Definition. For $\overline{x}, \overline{y} \in \overline{X}$, a **discrete walk** from $\overline{x}$ to $\overline{y}$ is a finite sequence

$$\alpha: \ x = x_0, x_0', x_2, x_2', x_3, x_3', \ldots, x_{n-1}, x_{n-1}', x_n, x_n' = y$$

such that for all k, $x_k \sim x_k'$ (i.e. $\overline{x_k} = \overline{x_k'}$). The **length** of α is

$$L(\alpha) = \sum_{k=0}^{n-1} d(x_k', x_{k+1}).$$

Lemma 4.6. If $a \sim x$ and $b \sim y$ then if there is a discrete walk from a to b there is a corresponding discrete walk from x to y of the same length.

We can now define the grasshopper metric.

Definition. Suppose that $\overline{x}, \overline{y} \in \overline{X}$. The **grasshopper distance** from $\overline{x}$ to $\overline{y}$ is

$$\overline{d}(\overline{x}, \overline{y}) = \inf\{L(\alpha) : \alpha \text{ is a discrete walk from } \overline{x} \text{ to } \overline{y}\}$$

Theorem 4.7. *The grasshopper distance $\overline{d}$ is a semi-metric on $\overline{X}$.*