Fall 2016/MA 314 HW 4: Try a triangle!

Geometry is a subject full of wonderful problems and interesting methods drawn from all quarters of mathematics. Some of the problems below are just intended to cement basic concepts or give practice with basic calculations, while others are intended to require substantial insight and even, occasionally, cleverness.

Remember that, although you are encouraged to work together, all of your write-ups must be your own (no copying someone else's solution - not even with minor wording changes.) **List the names of everyone you worked with on the HW!** You are encouraged to **not** look online for solutions - your time is better spent wrestling with the proof yourself or getting help from the professor, than squandering it online. Remember to follow the formatting requirements for your Also, the readings for the week are listed below for convenience, but you should look at the reading assignment webpage to see what the day-by-day reading (and reading response) schedule is.

1. Reading

- Watch the TED talk on crocheting hyperbolic planes. (It is 16 minutes 43 sec. long)
- Schwartz 2.7 - 2.8, 3.1 - 3.4
- Bonahon Chapters 4 and 5 (different parts will be studied to different levels of depth.)

2. To do:

(1) This problem is intended to give a taste of how ideas from hyperbolic geometry have had an impact in the rest of mathematics. In this case, the impact is in Group Theory. Without a background in group theory, it's difficult to go into too much detail. Suffice it to say, however, that in the 1980s, M. Gromov pinpointed the "slim triangles" property of hyperbolic space as key to many of the other properties of hyperbolic space.

Definition. Suppose that (X,d) is a metric space with a *path metric* d (i.e. we can measure the lengths of paths and every two points in X are joined by a path of shortest length (a geodesic.) For points $a,b \in X$, let $[a,b]$ denote a geodesic between a and b. We say that (X,d) is a δ-hyperbolic (for $\delta \geq 0$) if for all $a,b,c \in X$, and for all $x \in [a,b]$, there exists $y \in [b,c] \cup [a,c]$ such that $d(x,y) < \delta$.

(a) Suppose that T is a tree where every edge has length 1. Explain why T is 0-hyperbolic.

(b) Explain why $\mathbb{E}^2$ is not δ-hyperbolic, for any $\delta \geq 0$.

(c) Prove that $\mathbb{H}^2$ is δ-hyperbolic for some $\delta > 0$. What's the smallest possible value of δ that you can find? (See me for a hint)

Here's the connection to group theory. Given a group G, a subset $S \subset G$ **generates** G if every element of G is the combination (using the group operation) of elements of S and their inverses. (So for example, $\text{Isom}(\mathbb{R}^2)$ is generated by the set of translations, reflections, and rotations.) Associated to every generating set for G is a graph called the **Cayley-graph** for G. It turns out that the property of having a Cayley graph which is δ-hyperbolic is an intrinsic property of the group and governs many important group-theoretic aspects. Not all groups are δ-hyperbolic for some δ, for instance $\mathbb{Z} \times \mathbb{Z}$.

(2) Work on your group projects!