Fall 2016/MA 314 HW 3:Do spherical cows have Moo-bius transformations?

Geometry is a subject full of wonderful problems and interesting methods drawn from all quarters of mathematics. Some of the problems below are just intended to cement basic concepts or give practice with basic calculations, while others are intended to require substantial insight and even, occasionally, cleverness.

Remember that, although you are encouraged to work together, all of your write-ups must be your own (no copying someone else's solution - not even with minor wording changes.) **List the names of everyone you worked with on the HW!** You are encouraged to **not** look online for solutions - your time is better spent wrestling with the proof yourself or getting help from the professor, than squandering it online. Remember to follow the formatting requirements for your Also, the readings for the week are listed below for convenience, but you should look at the reading assignment webpage to see what the day-by-day reading (and reading response) schedule is.

1. Reading

- Schwartz 10.4, 11.1 - 11.3
- Schwartz 10.7 - 10.8
- Bonahon 2.6 - 2.7.
- Read the blog essay Random Turtles in the Hyperbolic Plane by Danny Calegari.

2. To do

(Remember that you are welcome to use a computer to compute integrals.)

(1) Bonahon Exercise 2.4

(2) Bonahon Exercise 2.5 (this and the previous exercise, will give you some sense for the ways in which hyperbolic geometry is like euclidean geometry)

(3) Bonahon Exercise 2.15 (this is the hyperbolic analogue of the previous result for spherical triangles)

(4) (Adapted from Bonahon Exercises 2.13 and 2.14) The point of this exercise is to understand how discs in $\mathbb{H}^2$ behave.

   (a) Let O be the center of the disc model $\mathbb{B}^2$ of the hyperbolic plane. Let B be the open ball consisting of all points $x \in \mathbb{B}^2$ of hyperbolic distance strictly less than r from O. Show that B is equal to the set of points in $\mathbb{B}^2$ each of which is distance strictly less than $\tanh(r/2)$ from O. Thus, the hyperbolic disc centered at O is also a euclidean disc centered at O, but of a different radius.

   (b) Compute (in terms of r) the length (i.e. circumference) of the circle of radius r in $\mathbb{B}^2$ centered at O.

   (c) Show that every hyperbolic disc in $\mathbb{H}^2$ is also a euclidean disc. Hint: use (a), the isometry of Proposition 2.21, as well as Propositions 2.2 and 2.18 (of Bonahon).

(d) The hyperbolic area of $D \subset \mathbb{H}^2$ is, by definition,

$$A(D) = \iint_D \frac{1}{y^2}\, dx\, dy.$$

Show that isometries of $\mathbb{H}^2$ preserve area.

(e) The hyperbolic area of $D \subset \mathbb{B}^2$ is (by a calculation using Proposition 2.21)

$$A(D) = \iint_D \frac{4}{(1-x^2-y^2)^2}\, dx\, dy.$$

Prove that a disc of *hyperbolic* radius r centered at O has hyperbolic area

$$2\pi(\cosh r - 1) = 4\pi \sinh^2(r/2).$$

Use your previous work to conclude that this is valid for any disc of radius r in either $\mathbb{B}^2$ or $\mathbb{H}^2$. Also compare the ratios Area/radius and Area/circumference and circumference/radius for the euclidean and hyperbolic planes.