MA 274: Exam 2 Study Guide

Here are some suggestions for what and how to study:

(1) Here are some results you should be especially sure to know how to prove. You should also think about ways these problems might be varied. And you should study other problems too **including the problems on the pair-review project**.

(a) The compositions of injections/surjections/bijections is a an injection/surjection/bijection.

(b) A function $f\colon X \to Y$ is a bijection if and only if there is a function $f^{-1}\colon X \to Y$ such that $f \circ f^{-1}(y) = y$ for all $y \in Y$ and $f^{-1} \circ f(x) = x$ for all $x \in X$.

Solution: Assume, first, that $f\colon X \to Y$ is a bijection. For each, $y \in Y$, let $f^{-1}(y)$ denote the unique element of X such that $f(f^{-1}(y)) = y$. Since f is surjective there is such an element and since f is injective, it is unique. Then $f^{-1}\colon Y \to X$ is a function as for each element $y \in Y$, there is a unique $x \in X$ with $f^{-1}(y) = x$. Furthermore, by definition, for all $y \in Y$, $f \circ f^{-1}(y) = f(f^{-1}(y)) = y$. Consequently, $f \circ f^{-1}$ is the identity function on Y. Suppose $x \in X$. Then $f^{-1}(f(x))$ is the unique x' such that $f(x') = f(x)$. In particular, $x' = x$ and so for all $x \in X$, $f^{-1} \circ f(x) = f^{-1}(f(x)) = x$. Thus, $f^{-1} \circ f$ is the identity function on X. Consequently, f^{-1} and f are inverse functions, as desired.

Now, suppose that $f\colon X \to Y$ has an inverse function $f^{-1}\colon Y \to X$. We will show that f is injective and surjective and is, therefore, a bijection. Suppose that $f(a) = f(b)$. Since f^{-1} is a function, we have

$$f^{-1}(f(a)) = f^{-1}(f(b)).$$

By the definition of inverse function, we have $a = b$. Thus, f is injective.

Now suppose that $y \in Y$. Let $x = f^{-1}(y)$. It exists since f^{-1} is a function. Consequently, $f(x) = f(f^{-1}(y)) = y$ since f and f^{-1} are inverses. Thus, for all $y \in Y$, there exists $x \in X$ with $f(x) = y$. Therefore, f is a bijection. $\square$

(c) Basic proofs by induction (see text, homework, and class notes)

(d) If X is a set with n elements, then every permutation (i.e. bijection $X \to X$) is the composition of transpositions. (Theorem 6.1.7 - we did this in class)

(e) Euler's theorem for planar graphs: If G is a finite, planar, non-empty, connected graph with $V(G)$ vertices, $E(G)$ edges, and $F(G)$ faces, then $V(G)-E(G)+F(G)=2$. (This is Theorem 6.2.4).

(f) The Well-Ordering Principle

Solution: Suppose that $S \subset \mathbb{N}$ is a set without a least element. We prove that $S = \varnothing$. That is, we prove that for all $n \in \mathbb{N}$, $n \notin S$. We use induction.

Consider $n = 1$. Since 1 is the least element of $\mathbb{N}$ and since $S \subset \mathbb{N}$, if $1 \in S$ then 1 is the least element of S, contradicting our initial assumption. Thus $1 \notin S$.

Suppose, therefore, that none of $1,\dots,k$ are elements of S. The number $k+1$ is the least element of $\mathbb{N}$ which is bigger than k, and so if $k+1 \in S$, then it would be the least element of S (as none of $1,\dots,k$ are elements of S.) Since S does not have a least element, $k+1 \notin S$.

Consequently, by induction, for all $n \in \mathbb{N}$, $n \notin S$. Hence, $S = \varnothing$. □

(g) Every integer greater than one is a multiple of a prime number.

Solution: We prove that every $n \in \mathbb{N}$ with $n \geq 2$ is a multiple of a prime number by induction. Suppose that $n = 2$. Then n is prime and is a multiple of itself, so the result holds.

Assume, therefore, that there is a $k \in \mathbb{N}$ with $k \geq 2$ and with the property that for all j with $2 \leq j \leq k$ the number j is a multiple of a prime. We show that $k+1$ is a multiple of a prime.

If $k+1$ is itself prime then it is a multiple of itself and is, therefore, a multiple of a prime. If $k+1$ is not prime, then, by the definition of prime, there exist $a,b \in \mathbb{N}$ such that neither a nor b are 1 or $k+1$ but $k+1 = ab$. Since $b \geq 2$, we have $a \leq k$. Thus, by our inductive hypothesis, a is a multiple of a prime. That is, there exists $m \in \mathbb{N}$ and a prime $p \in \mathbb{N}$ such that $a = mp$. Thus, $k+1 = mpb$ and so $k+1$ is also a multiple of a prime.

By induction, every natural number at least 2 is a multiple of a prime. □

(h) Every fraction can be written in lowest terms.

Solution: Suppose that $r \in \mathbb{Q}$ and that $r \geq 0$. Let

$$S = \{a \in \mathbb{N} \cup \{0\} : \exists b \in \mathbb{N} \text{ with } r = a/b.\}$$

By the definition of $\mathbb{Q}$, the set $S \neq \varnothing$. If $0 \in S$, then $r = 0 = 0/1$ is written in lowest terms since the only factors of 1 are ± 1. If $0 \notin S$, then $S \subset \mathbb{N}$. By the well-ordering principle, there exists an element

$a \in S$ which is the least element of S. By the definition of S, there exists $b \in \mathbb{N}$ such that $r = a/b$.

We claim that a and b have no common factors except ± 1. If some negative integer is a factor of both a and b, then its absolute value is a postive integer which is a factor of both a and b. It suffices to show that a and b have no common positive factors except 1. Suppose, therefore, that both a and b are multiples of $m \in \mathbb{N}$. Then there exist $k, \ell \in \mathbb{N}$ such that $a = mk$ and $b = m\ell$. Consequently

$$r = a/b = (mk)/(m\ell) = k/\ell.$$

Thus, $k \in S$. By the properties of multiplication, $k \leq a$. Since a is the least element of S, we have $k = a$. Thus, $\ell = b$ and $m = 1$ as desired. $\square$.

(i) If A contains an injective sequence then there is a proper subset $B \subset A$ and a bijection $f\colon A \to B$.

Solution: Let (a_n) be an injective sequence in A and let $B = A \setminus \{a_1\}$. For $a \in A$, define

$$f(a) = \begin{cases} a_{n+1} & \text{if } \exists n \in \mathbb{N} \text{ s.t. } a = a_n \\ a & \text{if } \forall n \in \mathbb{N}, a \neq a_n \end{cases}$$

Since (a_n) is an injective sequence, $f\colon A \to b$ is a well-defined function. It is a bijection it has an inverse $f^{-1}\colon B \to A$ defined by:

$$f^{-1}(b) = \begin{cases} a_{n-1} & \text{if } \exists n \in \mathbb{N} \setminus \{1\} \text{ s.t. } b = a_n \\ b & \text{if } \forall n \in \mathbb{N} \setminus \{1\}, b \neq a_n. \end{cases}$$

(j) If there is a surjection $f\colon X \to Y$ then there is an injection $g\colon Y \to X$.

Solution: Suppose that $f\colon X \to Y$ is a surjection. By the Axiom of Choice, there exists a subset $A \subset X$ such that the restriction $f|_A\colon A \to Y$ is a bijection. Let $g\colon Y \to A$ be its inverse. The function g is a bijection and extending its codomain to be all of X, preserves the fact that it is an injection, though perhaps losing surjectivity. Hence, there is an injection $g\colon Y \to X$.

(k) If X is an infinite set then there is an injective sequence in X.

(l) $\operatorname{card} \mathbb{N} \times \mathbb{N} = \operatorname{card} \mathbb{N}$ (the Cantor Snake)

(m) The rationals are countable

(n) If X is countable and if $A \subset X$, then A is countable.

(o) If there are bijections $X \to \{1, \ldots, n\}$ and $X \to \{1, \ldots, m\}$ for some $n, m \in \mathbb{N}$. Then $n = m$. (Be careful: this is harder than it looks!)

Solution: We prove that for all $n \in \mathbb{N}$ and for all $m \in \mathbb{N}$, if there is a bijection $f\colon X \to \{1,\dots,n\}$ and a bijection $g\colon X \to \{1,\dots,m\}$, then $n = m$. We induct on n.

If $n = 1$, then $\{1\}$ has a unique element. Since $f\colon X \to \{1\}$ is surjective, $X \neq \varnothing$. If $a, b \in X$, then $f(a) = f(b) = 1$. Since f is injective, this implies $a = b$. Hence X has a unique element. The function $g\colon X \to \{1,\dots,m\}$ is a bijection and so $g^{-1}\colon \{1,\dots,m\} \to X$ is a bijection. Hence, $g^{-1}(1) = g^{-1}(m)$ since X has a unique element and g^{-1} is surjective. Since g^{-1} is injective, $1 = m$. Thus, $n = 1 = m$ as desired.

Assume, therefore, that for all sets X, if there is some $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, if there is a bijection $f\colon X \to \{1,\dots,k\}$ and a bijection $g\colon X \to \{1,\dots,m\}$ then $k = m$. We prove that for all sets X' and for all $m' \in \mathbb{N}$ if there is a bijection $f'\colon X' \to \{1,\dots,k+1\}$ and a bijection $g'\colon X' \to \{1,\dots,m'\}$ then $k+1 = m'$.

Let X' be a set and $m' \in \mathbb{N}$ be such that there is a bijection $f'\colon X' \to \{1,\dots,k+1\}$ and a bijection $g'\colon X' \to \{1,\dots,m'\}$. Let $X = X' \setminus \{f^{-1}(k+1)\}$. Then the restriction $f = f'|_X$ is a bijection $f\colon X \to \{1,\dots,k\}$. Define $g\colon X \to \{1,\dots,m'-1\}$ be defined as follows:

$$g(x) = \begin{cases} g'(x) & \text{if } g'(x) < g'(f^{-1}(k+1)) \\ g'(x) - 1 & \text{if } g'(x) > g'(f^{-1}(k+1)). \end{cases}$$

The function g has inverse defined, for all $j \in \{1,\dots,m'-1\}$ by

$$g^{-1}(j) = \begin{cases} g'^{-1}(j) & j < g'(f^{-1}(k+1)) \\ g'^{-1}(j+1) & j \geq g'(f^{-1}(k+1)). \end{cases}$$

Thus, g is a bijection. By the inductive hypothesis, $k = m' - 1$. Consequently, $k + 1 = m'$, as desired. By induction the result holds. $\square$.

(p) The interval $(0,1)$ and the set $\mathbb{R}$ are uncountable

(q) For every set X, $\operatorname{card} X < \operatorname{card} \mathscr{P}(X)$.

(r) $\operatorname{card} \mathbb{R} = \operatorname{card} \mathscr{P}(\mathbb{N})$.

(s) Let $\mathscr{F}$ be the set of functions from $X \to \{0,1\}$. Then $\operatorname{card}(\mathscr{P}(X)) = \operatorname{card} \mathscr{F}$.

Solution: We construct a bijection $h\colon \mathscr{P}(X) \to \mathscr{F}$. For a set $A \subset X$ (i.e. an element of $\mathscr{P}(X)$) define $h(A)$ to be the function $h(A)\colon X \to \{0,1\}$ by declaring, for all $x \in X$,

$$h(A)(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Thus, $h(A) \in \mathscr{F}$ and so $h\colon \mathscr{P}(X) \to \mathscr{F}$ is a function. It has an inverse h^{-1} defined as follows. Suppose that $f\colon X \to \{0,1\}$ is an element of $\mathscr{F}$. Define $h^{-1}(f) = \{x \in X : f(x) = 1\}$. Clearly, $h^{-1}(f) \in \mathscr{P}(X)$. It is easily verified that h and h^{-1} are inverses. Hence, h is a bijection. Consequently, $\operatorname{card}\mathscr{P}(X) = \operatorname{card}\mathscr{F}$. $\square$

(2) Here are a few problems you haven't done before (solutions will be posted on the Tuesday before the exam):

(a) Suppose that A and B are both countable sets. Then $A \cup B$ is countable.
Solution: Since A is countable, there exists an injection $f\colon A \to \mathbb{N}$. (If A is infinite, we could take f to be a bijection, but this is not necessary.) Similarly, there is an injection $g\colon B \to \mathbb{N}$. Define a function $h\colon A \cup B \to \mathbb{N}$ by

$$h(x) = \begin{cases} 2f(x) & x \in A \\ 2g(x) - 1 & x \in B \setminus A. \end{cases}$$

Notice that the function is well defined since A and $B \setminus A$ are disjoint. We show that it is an injection. Assume that $h(x_1) = h(x_2)$. If this number is even, then both x_1 and x_2 are in A and so, since f is an injection $x_1 = x_2$. Similarly, if $h(x_1) = h(x_2)$ is odd, then both x_1 and x_2 are in $B \setminus A$. Since g is an injection we again have $x_1 = x_2$. Consequently, h is an injection.
Since there is an injection from $A \cup B$ to $\mathbb{N}$, there is a subset $X \subset \mathbb{N}$ such that $\operatorname{card}(A \cup B) = \operatorname{card} X$. Since X is countable, $A \cup B$ must be as well. $\square$.

(b) If $\operatorname{card} X = \operatorname{card} Y$ and $\operatorname{card} Y = \operatorname{card} Z$ then $\operatorname{card} X = \operatorname{card} Z$. (remember the precise definitions!)
Solutions: Assume $\operatorname{card} X = \operatorname{card} Y$ and $\operatorname{card} Y = \operatorname{card} Z$. Since $\operatorname{card} X = \operatorname{card} Y$, there exists a bijection $f\colon X \to Y$. Since $\operatorname{card} Y = \operatorname{card} Z$, there exists a bijection $g\colon Y \to Z$. Then by an earlier theorem, $g \circ f \circ X \to Z$ is a bijection and so $\operatorname{card} X = \operatorname{card} Z$.

(c) If $\operatorname{card} X = \operatorname{card} Y$ then $\operatorname{card} Y = \operatorname{card} X$.
Solution: Assume $\operatorname{card} X = \operatorname{card} Y$. By definition, there is a bijection $f\colon X \to Y$. By a previous thorem, there is an inverse function $f^{-1}\colon Y \to X$. Since f is the inverse function to f^{-1}, f^{-1} is also a bijection. Hence $\operatorname{card} Y = \operatorname{card} X$.

(d) Let $A = \{a\}$ be a set with a single element and let X be any set. Let $\mathscr{F}$ be the set of functions from A to X. Prove that $\operatorname{card}\mathscr{F} = \operatorname{card} X$.
Solution: We construct a bijection $h\colon X \to \mathscr{F}$. For $x \in X$, let $f_x \in \mathscr{F}$ be the function defined by

$$f_x(a) = x.$$

Then set $h(x) = f_x$. Then $h\colon X \to \mathscr{F}$ is a function.
Suppose that $g \in \mathscr{F}$. Define $h^{-1}(g) = g(a)$. Then $h^{-1}\colon \mathscr{F} \to X$ is a function.

We show that h and h^{-1} are inverses of each other. Let $x \in X$. Then $h^{-1}(h(x)) = h^{-1}(f_x) = f_x(a)$. This is equal to x by definition. Now suppose that $g \in \mathscr{F}$. Then $h(h^{-1}(g)) = h(g(a)) = f_{g(a)}$. But $f_{g(a)}$ is the function with the property that $f_{g(a)}(a) = g(a)$. Since $f_{g(a)}$ and g have the same domain, codomain, and take the same values on all elements of A, they are equal. That is, $h(h^{-1}(g)) = g$.
Thus, h and h^{-1} are inverses. $\square$

(e) Suppose that $A = \{a_1, a_2\}$ is a set with exactly two elements and let X be any set. Let $\mathscr{F}$ be the set of functions from A to X. Prove that $\operatorname{card}\mathscr{F} = \operatorname{card} X \times X$.

Solution: We construct a bijection $h \colon X \times X \to \mathscr{F}$. For $(x_1, x_2) \in X \times X$, define $h(x_1, x_2)$ to be the function on A defined by:

$$h(x_1,x_2)(a) = \begin{cases} x_1 & a = a_1 \\ x_2 & a = a_2 \end{cases}$$

for all $a \in A = \{a_1, a_2\}$.
We show that h is a bijection by producing an inverse. Define $h^{-1} \colon \mathscr{F} \to X \times X$ by, for each $f \in \mathscr{F}$, declaring

$$h^{-1}(f) = (f(a_1), f(a_2)).$$

Observe that $h^{-1}(h(x_1,x_2)) = (x_1,x_2)$ and that $h(h^{-1}(f))$ is the function which, for $a \in A$, takes the value of $f(a_1)$ when $a = a_1$ and $f(a_2)$ when $a = a_2$. In other words, $h(h^{-1}(f)) = f$. Thus, h is a bijection and so $\operatorname{card}\mathscr{F} = \operatorname{card} X \times X$. $\square$.