MA 331: Topological Dimension

The following discussion is based on that in Prasolov's book *Elements of Combinatorial and Differential Topology*.

Definition 1. If $\mathscr{U}$ is an open cover of $A \subset \mathbb{R}^n$ then the Lebesgue number $\Delta(\mathscr{U})$ of $\mathscr{U}$ is the least upper bound of

$$\{\delta \geq 0 : \forall B \subset A \text{ with } \operatorname{diam}(B) \leq \delta, \exists U \in \mathscr{U} \text{ s.t.} B \subset U\}\}$$

Lemma 2 (Lebesgue Covering Lemma)**.** Suppose that $A \subset \mathbb{R}^n$ is non-empty and compact and that $\mathscr{U}$ is an open cover of A. Then $\Delta(\mathscr{U}) > 0$.

Proof. Let $\{U_1, \ldots, U_k\}$ be a finite subcover of $\mathscr{U}$. Define $f_i \colon A \to \mathbb{R}$ by

$$f_i(x) = d(x, A \setminus U_i).$$

Since $A \setminus U_i$ is a closed subset of a compact set, it is compact. Thus f_i is well-defined. We have $f_i(x) = 0$ exactly when $x \notin U_i$. Let $f = \max(f_1, \ldots, f_k)$. Observe that $f \colon A \to \mathbb{R}$ is continuous and that for all $a \in A$, $f(a) > 0$. Thus, $f(A) \subset \mathbb{R}$ is a compact subset which does not contain 0. Consequently, there exists δ with $0 < \delta < \min f(A)$. Thus, for all $a \in A$, there exists i such that $d(a, A \setminus U_i) > \delta$. Hence, $A \cap B_\delta(a) \subset U_i$. If $B \subset A$ is any set of diameter less than $\delta/2$, and if $b \in B$, then $B \subset B_\delta(b)$. $\square$

Definition 3 (Lebesgue)**.** Suppose that $\mathscr{U}$ is a finite *closed* cover of a compact set $X \subset \mathbb{R}^n$. Define the **order** $\Omega(\mathscr{U})$ of $\mathscr{U}$ to be

$$m \in \mathbb{Z} : \quad \begin{array}{l} \exists x \in X \text{ s.t. } x \text{ belongs to } m \text{ elements of } \mathscr{U} \\ \text{and no element of } X \text{ belongs to } m+1 \text{ elements of } \mathscr{U} \end{array}$$

The **topological dimension** of X is equal to k if k is the least non-negative integer such that for all $\varepsilon > 0$, there exists a finite cover of X by closed sets of diameter less than ε and which has order $k+1$.

Definition 4. An n**-dimensional simplex** is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. (An **affine subspace** of $\mathbb{R}^n$ is the translation of a vector subspace of $\mathbb{R}^n$. Points are **affinely independent** if they do not lie in an $(n-1)$-dimension affine subspace of $\mathbb{R}^n$. The convex hull of a set of points is the smallest convex set containing those points.)

A 2-dimensional simplex is a (solid) triangle. A 3-dimensional simplex is a solid tetrahedron.

Theorem 5 (Dimension of a simplex)**.** The topological dimension of an n-simplex is n.

Proof. Let $\mathscr{U} = \{U_1, \dots, U_m\}$ be a finite closed cover of an n-simplex A.

Claim 1: If the diameter of each element of $\mathscr{U}$ is sufficiently small, then the **order** of $\mathscr{U}$ is $n+1$.

Let $F_0, \dots, F_n$ be the $(n-1)$-dimensional faces of A and let a_i be the vertex opposite F_i. The open cover $\mathscr{F}$ consisting of the sets $A \setminus F_i$ are an open cover of A. Let ε be the Lebesgue number of this cover. Suppose that each set in $\mathscr{U}$ has diameter less than ε. By the Lebesgue covering lemma, each U_j does not intersect some F_{n_j}. Also, if v is a vertex of A then it belongs to some U_v and this U_v is disjoint from the other vertices of A.

Let $\phi : \mathscr{U} \to \{F_1, \dots, F_n$ be a bijection such that for each $U j \in \mathscr{U}$, $\phi(U_j)$ is a face F_{n_j} with $F_{n_j} \cap U_j = \varnothing$. For each $k \in \{0, \dots, n\}$ let A_k be the union of the $U \in \mathscr{U}$ for which $\phi(U) = F_k$. Sperner's lemma and the sequential compactness of A implies that there exists $x \in \bigcap_k A_k$.

(Hint: Label a point $y \in A$ by the least k for which $y \in A_k$. Given a triangulation of A, this is a Sperner labelling of the vertices. Taking Barycentric subdivisions and repeatedly applying Sperner's Lemma give us a sequence having a convergent subsequence, converging to our desired x.)

We conclude that there is an element of x in the intersection of $(n+1)$ elements of $\mathscr{U}$. Thus the order of $\mathscr{U}$ must be at least $(n+1)$.

We now construct an closed cover with each set having diameter at most $\varepsilon > 0$ so that the cover has order $n+1$. This will complete the lemma.

Take the $(m+1)$st barycentric subdivision of A and look at the *star* of the vertices of the of the vertex. This is a closed cover and has order $n+1$. □

Lemma 6. If $X \subset \mathbb{R}^n$ is compact and if $A \subset X$ is compact, then the topological dimension of A is at most that of X.

Lemma 7. If X and Y are homeomorphic compact subsets of $\mathbb{R}^n$ then their topological dimensions are equal.

Proof. Let $h\colon X \to Y$ be a homeomorphism. Observe that if $\mathscr{U}$ is a finite closed cover of X of order κ, then $h(\mathscr{U})$ is a finite closed cover of Y having order κ. To see this, recall that order is just defined by a count of points being in a certain number of the sets from the cover. Those properties are unchanged by a homeomorphism.

Consider the set of open balls of radius $\varepsilon > 0$ in Y. These form an open cover of Y and their pre-image in X forms an open cover of X. Let δ be the Lebesgue number of this open cover. By the definition of topological dimension, there is a closed cover $\mathscr{V}$ of X so that each set in $\mathscr{V}$ has diameter less than δ and $\Omega(\mathscr{V}) = \kappa_X + 1$, where κ_X is the topological dimension of X. If $V \in \mathscr{V}$ there is a $y \in Y$ such that

$$V \subset h^{-1}(B_\varepsilon(y)).$$

Thus the diameter of $h(V)$ is at most 2ε. Since $h(\mathscr{V})$ has order $\kappa_X + 1$, we see that $\kappa_Y \leq \kappa_X$ where κ_Y is the topogical dimension of Y. Switching the roles of X and Y in the preceding argument shows that $\kappa_X = \kappa_Y$. □

Theorem 8. If $m \neq n$ then no open subset of $\mathbb{R}^m$ is homeomorphic to an open subset of $\mathbb{R}^n$.

Proof. Suppose, to the contrary, that $h\colon\ U \to V \subset \mathbb{R}^n$ is just such a homeomorphism. The set U contains an m-simplex A, having topological dimension m. There is a simplex $B \subset \mathbb{R}^n$ such that the compact set $h(A) \subset B$. Thus, $m \leq n$. A similar argument shows that $n \leq m$ and so $n = m$. □