MA 331: Introduction to Homology

Throughout we work with finite dimensional vector spaces over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Such vector spaces are all of the form:

$$\mathbb{F}^n = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$$

Recall that every vector space has an element $0 = (0,0,0,\ldots,0)$. Much of what we do can be done in a similar spirit using $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{Q}$ in place of $\mathbb{F}$.

1. Chain Complexes

Suppose that we have a sequence of vector spaces (C_n) over $\mathbb{F}$ with linear transformations $\partial_n \colon C_n \to C_{n-1}$ as below:

$$\stackrel{\partial_{n+1}}{\to} C_n \stackrel{\partial_n}{\to} C_{n-1} \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_3}{\to} C_2 \stackrel{\partial_2}{\to} C_1 \stackrel{\partial_1}{\to} C_0 \stackrel{\partial_0}{\to} \{0\}$$

Such a sequence is a **chain complex** if for every n, $\partial_n \circ \partial_{n+1} = 0$. That is, following two arrows in a row puts you at 0. Equivalently, for every n, $\operatorname{im} \partial_{n+1} \subset \ker \partial_n$.

Example:

Let $C_2 = \mathbb{F}^2$ and $C_1 = \mathbb{F}^3$ and $C_0 = \mathbb{F}^2$. Let ∂_1 and ∂_2 be the linear transformations given by the following matrices:

$$[\partial_1] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$[\partial_2] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Observe that (since we are working modulo 2), the product $[\partial_2][\partial_1]$ is the zero matrix. Thus, we have a chain complex:

$$\{0\} \to C_2 \stackrel{\partial_2}{\to} C_1 \stackrel{\partial_1}{\to} C_0 \to \{0\}$$

2. Quotient Vector Spaces

Recall that a vector space is (informally) a set V and a way of making linear combinations of elements of V using scalars in $\mathbb{F}$. That is, if $a, b \in V$ and $k, l \in \mathbb{F}$ then $ka + lb \in V$. Suppose that V is a vector space and that $W \subset V$ is a subspace. Define an equivalence relation $\sim_W$ on V by declaring:

$$a \sim_W b \Leftrightarrow (a - b) \in W.$$

We let V/W denote the set of equivalence classes. So an element $[v] \in V/W$ is the set of all vectors in V that differ by an element of W. Saying the same thing in other words, we consider two vectors v_1 and v_2 to be “the same” (and write them as $[v_1]$ or $[v_2]$) if they differ by an element of W.

It turns out that V/W is itself a vector space with the following definition of addition and scalar multiplication (where $v, u \in V$ and $k \in \mathbb{F}$):

$$\begin{aligned} [v] + [u] &= [v+u] \\ k[v] &= [kv] \end{aligned}$$

Of course, it needs to be verified that these definitions are well-defined and make V/W into a vector space. But they do!

Example 1. For this example, we use vector spaces over $\mathbb{R}$ rather than $\mathbb{F}$.

Let $V = \mathbb{R}^2$ and $W = \{(x,y) \in V : y = x\}$. Observe that W is a line through the origin in V. In V/W, the class $[(a,b)]$ is the line parallel to W and passing through (a,b). Thus, V/W is 1-dimensional (and isomorphic to $\mathbb{R}$). In fact, the space is isomorphic to the orthogonal complement of W in V. The isomorphism takes a point (a,b) of $W^\perp$ to the class $[(a,b)] \in V/W$.

Example 2. Again we use vector spaces over $\mathbb{R}$. Let $V = \mathbb{R}^3$ and let W be the span of the vector $(1,1,1)$. Then, in V/W, the class

$$[(a,b,c)] = \{(a,b,c) + t(1,1,1) : t \in \mathbb{R}\}.$$

Once again these are lines and we can find a vector space isomorphism between V/W and the orthogonal complement of W in V (i.e. the plane perpendicular to V). Hence, V/W is 2-dimensional.

The phenomenon we observed in the two examples is general. For convenience, we return to working with vector spaces over $\mathbb{F}$.

Theorem 3. Assume that V is a vector space over $\mathbb{F}$ and $W \subset V$ is a subspace. Assume also that V has an inner product. Then the quotient vector space V/W is isomorphic to $W^\perp$. In particular, if V is finite-dimensional, then $\dim V/W = \dim V - \dim W$.

Proof. Let $\phi : W^\perp \to V/W$ be given by $\phi(u) = [u]$. We claim that ϕ is a vector space isomorphism.

Claim 1: ϕ is linear.

Let $u, u' \in W^\perp$ and $k, m \in \mathbb{F}$. Then

$$\phi(ku + mu') = [ku + mu'] = k[u] + m[u'] = k\phi(u) + m\phi(u').$$

Claim 2: ϕ is injective.

Suppose that $\phi(u) = \phi(u')$. Then $[u] = [u']$ which means that there is $w \in W$ such that $u' - u = w$. However, $W^\perp$ is a subspace of V and so $w \in W^\perp \cap W$. Hence, $w = 0$ and so $u = u'$.

Claim 3: ϕ is surjective.

Choose a basis $\mathscr{U}$ of $W^\perp$ and a basis $\mathscr{V}$ of W. Then (after indexing the elements) $\mathscr{U} \cup \mathscr{V}$ is a basis for V. Let $[x] \in V/W$ and write x as a linear combination of (finitely many) of the elements from $\mathscr{U} \cup \mathscr{V}$:

$$x = \sum_{v \in \mathscr{U} \cup \mathscr{V}} k_v v$$

where $k_v \in \mathbb{F}$ and all but finitely many of the k_v are equal to zero. Let $u = \sum_{v \in \mathscr{U}} k_v v$ (i.e. the projection of x onto $W^\perp$). Observe that $x - u \in W$, so $[x] = [u]$. Hence, $\phi(u) = [u] = [x]$, as desired. □

3. Homology Groups

Let

$$\mathscr{C} : \cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \{0\}$$

be a chain complex. We define the nth **homology group** $H_n(\mathscr{C})$ to be the quotient vector space $(\ker \partial_n)/(\operatorname{im} \partial_{n+1})$. The nth (mod 2) **Betti number** is the dimension of $H_n(\mathscr{C})$.

Example 4. We consider the example from section 1. Let $\mathscr{C}$ be the chain complex:

$$\{0\} \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to \{0\}$$

where $C_2 = \mathbb{F}^2$ and $C_1 = \mathbb{F}^3$ and $C_0 = \mathbb{F}^2$. The boundary maps ∂_1 and ∂_2 are the linear transformations given by the following matrices:

$$[\partial_1] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$[\partial_2] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Calculations show that $\ker \partial_2$ has the vector $(1,1)$ as a basis. The image of ∂_3 is just 0, so

$$H_2(\mathscr{C}) = (\ker \partial_2)/(\operatorname{im} \partial_3) = \{(t,t) : t \in \mathbb{F}\}/\{0\} = \{t[(1,1)] : t \in \mathbb{F}\}$$

so $\dim H_2(\mathscr{C}) = 1$.

Similarly, $\{(1,1,1)\}$ is a basis for $\ker \partial_1$ and also for $\operatorname{im} \partial_2$ so $H_1(\mathscr{C}) = \{0\}$. Hence, $\dim H_1(\mathscr{C}) = 0$.

Finally, $\ker \partial_0 = C_0$ and $\operatorname{im} \partial_1$ has a basis $\{(1,0),(0,1)\}$. Since this implies ∂_1 is surjective, again we have $H_0(\mathscr{C}) = \{0\}$.

We now explain how to define the homology groups of a simplicial complex K.

Let K be a finite simplicial complex. For each n, we let $C_n(K) = \mathbb{F}^m$ where m is the number of n-dimensional simplices in K. (If there are none, we take $C_n(K) = \{0\}$. We identify each of the standard basis vectors of $\mathbb{F}^n$ with an n-dimensional simplex of K. Thus, if σ and τ are both n-dimensional simplices, then $\sigma + \tau$ is the vector in $\mathbb{F}^m$ obtained by adding the standard basis vector corresponding to σ to the standard basis vector corresponding to τ. Recall that since we are working mod 2, $\sigma + \sigma = 0$.

For an n-dimensional simplex σ, we let $\partial_n \sigma$ be the sum of the $(n-1)$-dimensional faces of σ and extend ∂_n linearly to all of $C_n(K)$. Thus, $\partial_n \colon C_n(K) \to C_{n-1}(K)$ is a linear map.

Lemma 5. Let $C_n(K)$ and $C_{n-1}(K)$ be as above. Then $\partial_{n-1} \circ \partial_n = 0$.

Proof. Let σ be an n-dimensional simplex, with $(n-1)$-dimensional faces $\tau_0, \ldots, \tau_n$. For each τ_i, there is exactly one vertex v_i of σ not in τ_i and τ_i is the convex hull of the vertices of σ other than v_i.

For some fixed i, let $\kappa_0, \ldots, \kappa_{n-1}$ be the $(n-2)$-dimensional faces of τ_i. For each κ_j, there is a vertex $v_j \neq v_i$ such that κ_j is disjoint from v_j and v_i and is the convex hull of the vertices of σ other than v_i and v_j. Thus, κ_j is also an $(n-2)$-dimensional face of τ_j and is not an $(n-1)$-dimensional face of any τ_k with $k \neq i, j$.

We have:

$$\begin{aligned} \partial_{n-1} \circ \partial_n(\sigma) &= \partial_{n-1}(\tau_0 + \tau_1 + \cdots + \tau_n) \\ &= \partial_{n-1}(\tau_0) + \cdots + \partial_{n-1}(\tau_n). \end{aligned}$$

By the previous paragraph, each $(n-2)$-dimensional face κ_j of σ occuring in the sum above occurs exactly twice. Since we are working modulo 2,

$$\partial_{n-1} \circ \partial_n(\sigma) = 0.$$

Since this is true for each basis element σ of $C_n(K)$, we have $\partial_{n-1} \circ \partial_n = 0$. □

Thus, we have a chain complex

$$\mathscr{C}(K) : \cdots \stackrel{\partial_{n+1}}{\to} C_n(K) \stackrel{\partial_n}{\to} C_{n-1}(K) \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_3}{\to} C_2(K) \stackrel{\partial_2}{\to} C_1(K) \stackrel{\partial_1}{\to} C_0(K) \stackrel{\partial_0}{\to} \{0\}$$

and we define $H_n(K) = H_n(\mathscr{C}(K))$.

Lemma 6. Let K be a finite simplicial complex. Then $\beta_0 = \dim H_0(K)$ is the number of connected components of K.

Proof. By definition, $H_0(K) = \ker \partial_0 / \operatorname{im} \partial_1 = C_0(K) / \operatorname{im} \partial_1$. Consider two vertices v_1 and v_2. They are related if and only if $v_1 - v_2 = v_1 + v_2 \in \operatorname{im} \partial_2$. The sum $v_1 + v_2$ is in the image of ∂_2 if and only if there is an edge path $e_1, e_2, \ldots, e_m$ joining v_1 to v_2, for then

$$\partial_1(e_1 + e_2 + \ldots + e_m) = v_1 + v_2.$$

Thus in $C_0(K)/\operatorname{im}\partial_1$, we have $[v_1] = [v_2]$ if and only if there is an edge path joining v_1 to v_2. So $H_0(K)$ is generated by a vertex from each path component of K. Since K is a simplicial complex, its path components are precisely its components. □

We can also see that in graphs $\beta_1 = \dim H_1(K)$ measures the number of independent loops in the graph.

Lemma 7. Let G be a connected graph and let $T \subset G$ be a maximal tree. Then $\beta_1 = \dim H_1(K)$ is equal to the number of edges of $G \setminus T$.

Proof. We have $H_1(K) = \ker\partial_1/\operatorname{im}\partial_2$. Since K is a 1-dimensional simplicial complex, $\operatorname{im}\partial_2 = \{0\}$. A collection of edges $e_1,\dots,e_m$ is in $\ker\partial_1$ if and only if each vertex appearing as an endpoint of some e_i appears as the end point of an even number of the e_i. That is, the union $e_1 \cup \dots \cup e_m$ is a collection of (possibly non-simple) loops in K.

Choose a vertex $v \in T$ and for each vertex $w \in T$, let α_w be the unique edge path having no backtracking from v to w. Let $\overline{\alpha}_w$ be the same edge path, but traversed in the reverse direction. Let e be an edge of $G \setminus T$ with endpoints u and w. Then $\gamma_e = \overline{\alpha}_u + e + \alpha_w$ is an element of $\ker\partial_1$. We claim that the set $\{[\gamma_e] : e \subset G \setminus T\}$ generates $H_1(K)$.

To see this, let $e_1 + \ldots + e_m$ be the sum of the edges lying in loop α (each appearing exactly once). Possibly after renumbering, we may assume that $e_1,\dots,e_k$ lie in $G \setminus T$ and $e_{k+1},\dots,e_m$ lie in T. Then

$$\alpha + \gamma_{e_m} = e_1 + \ldots + e_{k-1} + e'_k + \ldots + e'_{m'}$$

where $e'_k,\dots,e'_{m'}$ is a collection of edges in T since we are working mod 2 and all edges of γ_{e_m} lie in T except e_m. Continuing in a similar vein, the sum

$$\alpha + \gamma_{e_1} + \cdots + \gamma_{e_m}$$

can be written as the sum of edges lying entirely in T. It is in the kernel of ∂_1 and so it forms a collection of loops lying in the tree T. Every edge of a loop in a tree must appear an even number of times and so $\alpha + \gamma_{e_1} + \cdots + \gamma_{e_m} = 0$. Hence,

$$\alpha = \gamma_{e_1} + \cdots + \gamma_{e_m}$$

and so $\{[\gamma_{e_1}],\dots,[\gamma_{e_m}]\}$ spans $H_1(K)$ as desired.

If $\{[\gamma_{e_1}],\dots,[\gamma_{e_m}]\}$ were linearly dependent, then some γ_{e_j} is the sum of some of the others. However, this is impossible as none of the others contains e_j. Thus, we have our basis.

□

We have a similar result for surfaces:

Lemma 8. Suppose that K is a simplicial complex with its geometric realization $|K|$ homeomorphic to a closed surface S. Then any embedded curve $\gamma \subset S$ which is the union of edges of K represents a non-trivial element of $H_1(S)$ if and only if it is non-separating.

Proof. Let γ be an edge loop in K. Then $\gamma \in \ker \partial_1$ since it is a closed loop. Since every edge of K is adjacent to exactly two triangles, splitting S along γ creates a surface S' with two copies of γ in its boundary. Since S is closed, $\partial S'$ consists exactly of two copies of γ. If γ is separating, let $T_1, \ldots, T_n$ be the triangles in one component of S'. Then $\partial_2(T_1 + \ldots + T_n) = \gamma$ and so $[\gamma] = 0 \in H_1(K)$. On the other hand, if γ is non-separating and if $T_1, \ldots, T_m$ are triangles with $\gamma = \partial_2(T_1 + \ldots + T_n)$ then $T_1 \cup \ldots \cup T_n$ must be a surface having boundary exactly equal to γ. However, there is no such surface for splitting S along γ, produces a surface $S' = T_1 \cup \ldots T_n$ with $\partial S'$ the union of two copies of γ. In which case $\partial_2(T_1 + \ldots + T_n) = 2\gamma = 0$. □

Finally, we consider the second homology group of a surface.

Lemma 9. Let K be a simplicial complex with geometric realization homeomorphic to a connected surface S. If $\partial S = \varnothing$, then $\beta_2 = \dim H_2(K) = 1$ and if $\partial S \neq \varnothing$, then $H_2(K) = \{0\}$.

Proof. Recall that $H_2(K) = \ker \partial_2 / \operatorname{im} \partial_3$. Since K is 2-dimensional, $\operatorname{im} \partial_3 = \{0\}$. Thus we need only identify those sums of triangles in K which are in the kernel of ∂_2. If a 2-chain $\sigma_1 + \ldots + \sigma_n$ (with each σ_i a triangle of K) lies in $\ker \partial_2$, then if an edge of K appears in $\sum \partial(\sigma_i)$, it must appear an even number of times. Since there is at most two triangles adjacent to each edge of K, this means that each edge appearing in the sum, must appear exactly twice. Hence, each edge appearing in the sum is adjacent to exactly two triangles. If $\partial S \neq \varnothing$, then no non-trivial 2-chain can lie in $\ker \partial_2$ and so $\ker \partial_2 = \{0\}$ implying $H_2(K) = \{0\}$. If $\partial S = \varnothing$, then any non-trivial 2-chain appearing in $\ker \partial_2$ must contain all the triangles of K and so there is exactly one such 2-chain. Hence $\ker \partial_2$ contains one-element – the whole surface. Thus, $H_2(K)$ is 1-dimensional, as desired. □

4. Invariance

The following theorem is quite challenging to prove and requires a new tool called "singular homology" - we don't go into it here.

Theorem 10 (BIG Theorem)**.** If K and L are finite simplicial complexes with $|K|$ and $|L|$ homeomorphic. Then for all n, $H_n(K)$ is isomorphic to $H_n(L)$.

This theorem implies that euler characteristic is a topological invariant. Recall that the euler characteristic of a simplicial complex is

$$\chi(K) = \sum_n (-1)^n (\text{\# of } n \text{ -dimensional simplices})$$

and that the nth (mod 2) Betti number is $\beta_n = \dim H_n(K)$.

Theorem 11 (Euler Characteristic)**.** $\chi(K) = \sum_n (-1)^n \beta_n$. Consequently, $\chi(K)$ is invariant under homeomorphism since the homology groups are.

Proof. Let $d_n = \dim C_n(K)$ and $k_n = \dim \ker \partial_n$ and $r_n = \dim \operatorname{im} \partial_n$. By Theorem 3, $\beta_n = k_n - r_{n+1}$. By the rank-nullity theorem from linear algebra, $d_n = k_n + r_n$. By the definition of $C_n(K)$, d_n is equal to the number of n-dimensional simplices in K. Consequently

$$\begin{aligned}
\chi(K) &= \textstyle\sum_n (-1)^n d_n \\
&= \textstyle\sum_n (-1)^n (k_n + r_n) \\
&= \textstyle\sum_n (-1)^n k_n + \sum_n (-1)^n r_n \\
&= \textstyle\sum_n (-1)^n k_n + r_0 + \sum_n (-1)^{n+1} r_{n+1} \\
&= r_0 + \textstyle\sum_n (-1)^n (k_n - r_{n+1}) \\
&= r_0 + \textstyle\sum_n (-1)^n \beta_n \\
&= \textstyle\sum_n (-1) \beta_n
\end{aligned}$$

where the last equality arises from the fact that $r_0 = 0$ as ∂_0 is the 0 function. □

5. INDUCED MAPS

Theorem 12. Suppose that K and L are simplicial complexes and $f\colon K \to L$ is a simplicial map. Then, for all n, there is an induced linear transformation

$$f_*\colon H_n(K) \to H_n(L)$$

Proof. We begin by defining a linear map $f_*\colon C_n(K) \to C_n(L)$ and we'll show that it gives rise to the desired linear map $f_*\colon H_n(K) \to H_n(L)$.

Recall that the generators of $C_n(K)$ are identified with the n-simplices of K. Let σ be an n-simplex. Since f is simplicial, $f(\sigma) = \tau$ is a simplex of dimension at most n in L. If τ is an n-simplex, define $f_*(\sigma) = \tau$ and if τ has dimension strictly less than n, let $f_*(\sigma) = 0 \in C_n(L)$. For the sum $\sigma_1 + \ldots + \sigma_k$ of n-simplices in $C_n(K)$, define $f_*(\sigma_1 + \cdots + \sigma_k) = f_*(\sigma_1) + \cdots + f_*(\sigma_k)$. It is evident that $f_*\colon C_n(K) \to C_n(L)$ is linear.

Consider the diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1}(K) & \xrightarrow{\partial_{n+1}} & C_n(K) & \xrightarrow{\partial_n} & C_{n-1}(K) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\cdots & \longrightarrow & C_{n+1}(L) & \xrightarrow{\partial'_{n+1}} & C_n(L) & \xrightarrow{\partial'_n} & C_{n-1}(L) & \longrightarrow & \cdots
\end{array}$$

It is not difficult to show, using the fact that f is simplicial, that it commutes.

Suppose, therefore, that $\sigma = \sigma_1 + \cdots + \sigma_k \in \ker \partial_n$, where each σ_i is an n-simplex of K. We wish to show that $f_*(\sigma) \in \ker \partial'_n$. Since $\sigma \in \ker \partial_n$, we have:

$$\partial'_n(f_*(\sigma)) = f_*(\partial_n(\sigma)) = f_*(0) = 0,$$

using the linearity of f_* and the commutativity of the diagram.

Now suppose that $\sigma, \tau \in [\sigma] \in H_n(K)$. We want to show that $f_*(\sigma), f_*(\tau) \in [f_*(\sigma)]$. By th definition of $H_n(K)$, there exists an $(n+1)$-dimensional chain $\omega \in C_{n+1}(K)$ such that $\partial_{n+1}(\omega) = \sigma - \tau$. By the commutativity of the diagram,

$$f_*(\sigma) - f_*(\tau) = (f_*(\sigma - \tau) = f_*(\partial_{n+1}(\omega)) = \partial'_{n+1}(f_*(\omega)).$$

Thus, $f_*(\sigma)$ and $f_*(\tau)$ differ by an element in the image of ∂'_{n+1}, as desired.

Consequently, the induced map $f_* \colon H_n(K) \to H_n(L)$ defined by $f_*([\sigma]) = [f_*(\sigma)]$ is well-defined. It is easy to check that it is linear. □

Taking for granted that all this works, even when we don't have a triangulation in sight (i.e. using singular homology), we can prove the Brouwer fixed point theorem, via the following theorem (which is important in its own right).

Theorem 13 (No retraction theorem)**.** If M is a compact, non-empty 3-manifold, then there is no map $r \colon M \to \partial M$ such that for all $x \in \partial M$, $r(x) = x$. (Such a map is called a retraction of M onto ∂M.

We make use of the following fact, which you are invited to ponder:

Fact: If X is a non-empty compact manifold, without boundary, of dimension m, then $H_m(X)$ is one-dimensional (over[1] $\mathbb{F}$).

Proof. Suppose, for a contradiction, that $r \colon M \to \partial M$ is a retraction of an n-dimensional manifold to its boundary. Since M is non-empty, ∂M cannot be empty.

Let $i \colon \partial M \to M$ be the map induced by inclusion. Then, by the definition of retraction,

$$r \circ i \colon \partial M \to \partial M$$

is the identity map. The identity map on a topological space induces the identity isomorphism on homology groups, so we have that

$$r_* \circ i_* = (r \circ i)_* \colon H_{n-1}(\partial M) \to H_{n-1}(\partial M)$$

is an isomorphism.

However, since, in M, ∂M bounds an n-manifold, and since the class $[\partial M]$ is a generator for $H_{n-1}(\partial M)$, the map

$$i_* \colon H_{n-1}(\partial M) \to H_{n-1}(M)$$

is the 0-map. Since $\partial M \neq \varnothing$, there is no way that $(r \circ i)_* = r_* \circ i_*$ can be the identity map on a 1-dimensional space. This contradiction implies that the map r cannot exist. □

[1] This is not true if we change the coefficient field, in which case the result depends on orientability

Observe that the requirement that M is compact is essential as there is a retraction from the upper half plane or $\mathbb{R}^2$ onto the x-axis.

We can now prove:

Theorem 14 (Brouwer Fixed Point Theorem)**.** Suppose that B is an n-dimensional ball. If $r\colon B \to B$ is continuous, then there exists $x \in B$ such that $f(x) = x$.

Proof. We prove this by contradiction. Suppose that $f\colon B \to B$ does not have a fixed point. For each $z \in B$, consider the ray emitting from $f(z)$ and passing through z. Since f does not have a fixed point, there is a unique such ray. Let $g(z) \in \partial B$ be the point where the ray intersects ∂B. The map $g\colon B \to \partial B$ is clearly continuous, since f is. It is also a retraction from B to ∂B which is impossible. $\square$