MA 331 HW 7: Keep an open mind – study topology!

1. Some definitions

Definition 1.1. Suppose that X is a topological space and that (x_n) is a sequence in X. We say that (x_n) **converges** to $x \in X$ if for every open set U with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we must have $x_n \in U$. (That is, the sequence converges to x if it is eventually contained in any open set containing x.)

Definition 1.2. Suppose that X has a topology $\mathscr{T}$. A subset $\mathscr{U} \subset \mathscr{P}(X)$ is a **basis** for $\mathscr{T}$ if every element of $\mathscr{T}$ is the union of elements from $\mathscr{U}$.

(Note that if $\mathscr{U}$ is a basis for a topology $\mathscr{T}$, then the intersection of any finite number of sets in $\mathscr{U}$ must be the union of sets in $\mathscr{U}$, so we don't have to worry about the finite intersection axiom.)

Definition 1.3. Suppose that $A \subset X$ and that X is a metric space. The **boundary** of A (denoted $\operatorname{Bdry} A$) is the set of points $x \in X$ such that every open set U in X with $x \in U$ has the property that there exists a point $a \in A$ and a point $b \in X \setminus A$ with $a, b \in U$.

It is possible to prove that the boundary of A is equal to the boundary of $X \setminus A$.

2. Problems

(1) A topological space $(X, \mathscr{T})$ is **Hausdorff** if for every $a, b \in X$ with $a \neq b$, there exist open sets U_a and U_b such that $a \in U_a$, and $b \in U_b$, and $U_a \cap U_b = \varnothing$. In Mendelson problem 1, page 74 you proved that a metrizable topology is Hausdorff.

  (a) Suppose that $(X, \mathscr{T})$ is a Hausdorff (but not necessarily metrizable) topological space. Suppose that (x_n) is a sequence in X which converges to a point $x \in X$. Prove that x is the *unique* limit of the sequence.

  (b) Give an example of a non-Hausdorff space $(X, \mathscr{T})$ and a sequence (x_n) which converges in X to two distinct points.

(2) Let X be an infinite set. Let $\mathscr{U}$ be the set of subsets of X whose complement is finite. Prove that $\mathscr{U}$ is a basis for a topology $\mathscr{T}$ on X. Is this topology neccesarily Hausdorff? (Be sure to prove the answer is yes, or give a counter-example to show the answer is no.)

(3) Let X be a topological space and let $A \subset X$. Is it the case that $\operatorname{Bdry}(\operatorname{Bdry}(A)) = \varnothing$? Either prove the answer is yes (no matter what X and A are) or give

a counterexample to show the answer is no. If the answer is no, can you figure out what the right statement about $\mathrm{Bdry}(\mathrm{Bdry}(A))$ is and prove it?

(4) Given topological spaces X and Y, a function $f\colon X \to Y$ is **continuous** if and only if for every open set $U \subset Y$, the set

$$f^{-1}(U) = \{x \in X | f(x) \in Y\}$$

is open.

(a) Prove that if X and Y are metric spaces then a function is continuous with this definition if and only if it is continuous with the definition involving metrics.

(b) Suppose that X, Y, Z are topological spaces (not necessarily metric spaces). Prove that if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then $g \circ f\colon X \to Z$ is continuous.