MA 331 Homework 6: Topology from the Top!

Please keep track of how long this assigment take you

1. Reading

(1) Read 3.1 and 3.2 from Mendelson. Pay close attention to the examples on page 72.

2. Problems

(1) Suppose that $X_1, \ldots, X_n$ are each sequentially compact metric spaces. Give a completely rigorous proof that the product

$$X_1 \times X_2 \times \cdots \times X_n$$

is sequentially compact.

(Remark: We have suppressed the notation for the metric on each of the spaces. The metric on the product is the maximum of the metrics on the X_i, as usual.)

(2) (*) Do problems 1, 4, 5 on page 74 of Mendelson. (You may need to recall DeMorgan's Laws for problem 4)

(3) Suppose that $(X, \mathscr{T})$ is a topological space and that $A \subset X$. Consider the set $\mathbb{U} = \{U \in \mathscr{T} : U \subset A\}$. That is, $\mathbb{U}$ is the set of all open subsets which are wholly contained in A. Prove that the union $\bigcup \mathbb{U}$ of all the sets in $\mathbb{U}$ is an element of $\mathbb{U}$. (We say that $\mathbb{U}$ has a "largest" element.)

(4) Suppose that $(X, \mathscr{T})$ is a topological space and that $A \subset X$. Consider the set

$$\mathbb{V} = \{V \in \mathscr{P}(X) : V^C \in \mathscr{T} \text{ and } A \subset V\}$$

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That is, $\mathbb{V}$ is the set of all closed subsets which contain A as a subset. Prove that the intersection $\bigcap \mathbb{V}$ of all the sets in $\mathbb{V}$ is an element of $\mathbb{V}$. (We say that $\mathbb{V}$ has a "smallest" element.)

(5) (*) Let $X = \{1, 2, 3\}$. List all the topologies on X.

(6) Let $X = \{1, 2, 3, 4\}$. List 8 different topologies on X.

(Last problem on next page.)

(7) (*) Consider a set X and let $\mathscr{T}_\alpha$ be a topology on X for all α in some index set Λ.

(a) Prove that $\mathscr{T} = \bigcap_{\alpha \in \Lambda} \mathscr{T}_\alpha$ is a topology on X.

(b) Give an example to show that if $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on a set X then $\mathscr{T}_1 \cup \mathscr{T}_2$ is not necessarily a topology on X.