MA 331 Homework 4: Taking it to the limit!

1. Reading

(1) Read Section 2.6 (page 52) of Mendelson. One idiosyncrasy is his definition of "open set". I would personally take the statement of Theorem 6.2 as the definition of "open set". Focus on Theorems 6.3 and 6.4 and the definition of "closed". We'll discuss these in class.

(2) Recall the definitions of inf (also called glb) and sup (also called lub):

Definition 1.1. Suppose that $A \subset \mathbb{R}$. A number $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$ is an **upper bound** for A if, for all $a \in A$, we have $a \leq \alpha$. The number α is the **supremum** of A (and we write $\alpha = \sup A$) if α is an upper bound for A and whenever x is an upper bound for A, we have $\alpha \leq x$. (That is, α is the *least upper bound* for A)

A number $\beta \in \mathbb{R} \cup \{-\infty, \infty\}$ is a **lower bound** for A if, for all $a \in A$, we have $\beta \leq a$. The number β is the **infimum** of A (and we write $\beta = \inf A$) if β is a lower bound for A and whenever x is an lower bound for A, we have $x \leq \beta$. (That is, β is the *greatest lower bound* for A)

Note that if $A = \varnothing$, then $\inf A = \infty$ and $\sup A = -\infty$.

Definition 1.2. Suppose that (X, d_X) is a metric space and that $A \subset X$. The subset A is **sequentially compact** if whenever (a_n) is a sequence in A, it has a subsequence (a_{n_k}) which converges to a point in A.

2. Problems

The point of today's problems are to prove the following theorem. For its proof you may use results discussed in class (such as Dedekind's theorem). Please don't look up the proof of this theorem elsewhere.

Theorem. The subset $[0,1]$ of $\mathbb{R}$ is sequentially compact.

Proof. Assume, for a contradiction, that (a_n) is a sequence in $[0,1]$ which does not have a convergent subsequence.

(1) Prove that the set of points $\{a_n\}$ is infinite.

(As a consequence, we may assume (by replacing (a_n) with a subsequence) that (a_n) has distinct terms.)

(2) Let $A_1 = \{a_n : n \geq 1\}$ and let $\alpha_1 = \sup A_1$. Prove that $\alpha_1 \in A_1$.

(As a consequence, there is $n_1 \in \mathbb{N}$ such that $a_{n_1} = \alpha_1$)

(3) Let $A_2 = \{a_n : n > n_1\}$. Let $\alpha_2 = \sup A_2$. Prove that $\alpha_2 \leq \alpha_1$.

(4) Prove that $\alpha_2 \in A_2$.

(5) Explain how to define, for all $k \in \mathbb{N}$, sets A_k such that for $\alpha_k = \sup A_k$, we have

$$\alpha_1 \geq \alpha_2 \geq \dots$$

and $\alpha_k \in A_k$.

(6) Use points α_k to define a subsequence of (a_n) which converges. Explain why it converges to a point in A and explain why this finishes the proof.

$\square$