MA 331 Homework 3 Hint:

(3) (Challenging!) Suppose that (X, d_X), (Y, d_Y) and (Z, d_Z) are all metric spaces. Give $X \times Y$ the metric $d_P = \max(d_X, d_Y)$. Let

$$\begin{aligned} \pi_X &: X \times Y \to X \\ \pi_Y &: X \times Y \to Y \end{aligned}$$

be the projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$ respectively. Suppose that $f\colon Z \to X \times Y$ is a function such that the compositions

$$\pi_X \circ f\colon Z \to X \text{ and } \pi_Y \circ f\colon Z \to Y$$

are both continuous. Prove that f is continuous.

Proof hint. Let $z_0 \in Z$ and let $\varepsilon > 0$. We must show that there exists $\delta > 0$ such that $f(B_\delta(z_0)) \subset B_\varepsilon(f(z_0))$. For convenience, let $(x_0, y_0) = f(z_0)$.

Since $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} \pi_X \circ f(B_{\delta_1}(z_0)) &\subset B_\varepsilon(x_0) \\ \pi_Y \circ f(B_{\delta_2}(z_0)) &\subset B_\varepsilon(y_0) \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$. We must show that for all $z \in B_\delta(z_0)$ we have $f(z) \in B_\varepsilon(x_0, y_0)$.

$\langle$ Do some work $\rangle$

Thus, we conclude that $f(B_\delta(z_0)) \subset B_\varepsilon(f(z_0))$. Hence f is continuous at z_0. Since z_0 was arbitrary, we conclude that f is continuous. $\square$