MA 331 Homework 3:

1. Reading

Most homework assignments will require you to read about material we have not yet covered in class. Doing this reading will help you gain independence as a mathematician and will also help us make effective use of class time. If you do the reading in advance of class (as you should!) you'll be better prepared to ask questions, have misunderstandings cleared up, and absorb more of the subtleties of the subject.

(1) Skim Section 2.4 of Mendelson. Pay particular attention to the definition of "neighborhood". We covered most of this in class.

(2) Read Section 2.5 of Mendelson. Pay particular attention to Corollary 5.3 and Theorem 5.4. The latter is an extremely important theorem.

2. Problems

(1) (*) Be prepared to present the proof of Theorem 5.4 from Mendelson Section 3.5

(2) Give $\mathbb{R}$ the euclidean metric. Let (X, d_X) be a metric space and suppose that

$$f\colon (X, d_X) \to \mathbb{R}$$

and

$$g\colon (X, d_X) \to \mathbb{R}$$

are continuous functions. Let $\kappa \in \mathbb{R}$.

Define functions $f+g\colon X \to \mathbb{R}$ and $\kappa f\colon X \to \mathbb{R}$ as follows:

$$\begin{aligned} f+g(x) &= f(x)+g(x) \\ (\kappa f)(x) &= \kappa f(x). \end{aligned}$$

Prove that $f+g$ and κf are continuous.

Hint for $f+g$: Let $x_0 \in X$ and assume that $\varepsilon > 0$ has been given. You must show that there exists a $\delta > 0$ such that for any $x \in X$ with $d_X(x, x_0) < \delta$ we have $|f(x)+g(x)-(f(x_0)+g(x_0))| < \varepsilon$. Since f is continuous, there is a δ_1 such that if $d_X(x, x_0) < \delta_1$ then $|f(x)-f(x_0)| < \varepsilon/2$. Likewise, since g is continuous, there is a δ_2 such that if $d_X(x, x_0) < \delta_2$ then $|g(x)-g(x_0)| < \varepsilon/2$. Use those facts and the triangle inequality to produce the desired δ.

(3) (Challenging!) Suppose that (X, d_X), (Y, d_Y) and (Z, d_Z) are all metric spaces. Give $X \times Y$ the metric $d_P = \max(d_X, d_Y)$. Let

$$\begin{aligned} \pi_X &: X \times Y \to X \\ \pi_Y &: X \times Y \to Y \end{aligned}$$

be the projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$ respectively. Suppose that $f\colon Z \to X \times Y$ is a function such that the compositions

$$\pi_X \circ f\colon Z \to X \text{ and } \pi_Y \circ f\colon Z \to Y$$

are both continuous. Prove that f is continuous.

If you would like a hint for the problem, follow this link. Be sure to note on your assignment that you used the hint.