MA 331 HW 22

(1) Let X be a genus 1 compact surface with a single boundary component (i.e. a torus with an open disc removed). Let Y be any compact surface with a single boundary component. Let S^1 be the unit circle in $\mathbb{R}^2$, oriented counter-clockwise. Let $i_X : \partial X \to S^1$ and $i_Y : \partial Y \to S^1$ be homeomorphisms. Give ∂X and ∂Y orientations so that i_X and i_Y are orientation preserving.

(a) Prove that there is a homeomorphism $h\colon X \to X$ such that $i_X \circ h|_{\partial X}$ is orientation reversing. (Hint: Use the fact that you know what X "looks like" and draw a picture to explain your map.

(b) Let $f\colon \partial X \to \partial Y$ be a homeomorphism which is orientation preserving. (That is, $i_Y \circ f \circ i_X^{-1} \colon S^1 \to S^1$ takes the counter-clockwise orientation to the counterclockwise orientation.) Let $g\colon \partial X \to \partial Y$ be a homeomorphism which is orientation reversing. Prove that $X \cup_f Y$ is homeomorphic to $X \cup_g Y$. (Hint: you will need to use the fact that if two gluing maps are isotopic, then the results of gluing are homeomorphic. You will also need to use the fact that every homeomorphism of S^1 is isotopic to either $z \mapsto z$ or $z \mapsto \bar{z}$.)

The remaining problems concern groups. Recall that a group is a set G with a binary operation (with the symbol usually omitted) so that:

(G1) For all $a, b \in G$, we have $ab \in G$.

(G2) There exists $\mathbf{1} \in G$ such that for all $a \in G$, $a\mathbf{1} = \mathbf{1}a = a$.

(G3) For all $a \in G$, there exists $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = \mathbf{1}$.

(G4) For all $a, b, c \in G$, we have $a(bc) = (ab)c$.

(2) Prove that the set $\mathrm{Homeo}(X)$ of homeomorphisms of a topological space X to itself is a group with the operation of function composition.

(3) Prove that the set of bijections of $\mathbb{R}^2$ to itself of the form:

$$f(x, y) = (x + n, y + m)$$

for some $n, m \in \mathbb{Z}$ is a group with the operation of function composition.

(4) Let a, b be symbols and let F_2 be the set of all finite "words" (i.e. finite sequences) in the symbols a, a^{-1}, b, b^{-1} where we consider two words w and w' to be the same if they differ by a finite sequence of moves of the following form:

- Removing or inserting adjacent pairs of the form aa^{-1}, bb^{-1}, $a^{-1}a$, or $b^{-1}b$.

We consider the "empty word" to be an element of F_2. Show that F_2 is a group under the operation of concatenation. That is if $w, w' \in F_2$ then the word ww' is obtained by writing all the letters of w followed by all the letters of w'.