MA 331 HW 21

This homework assignment will guide you through two applications of Sperner's Lemma.

1. TOPOLOGICAL DIMENSION OF AN n-SIMPLEX.

In class we stated the result that the topological dimension of an n-simplex is n, but did not prove it. The following steps will guide you through the proof. You should review the definitions of topological dimension, order, Lebesgue number before beginnning this.

Let A be an n-dimensional simplex with vertices $a_0, \ldots, a_n$. Remember that each non-empty subset of the vertices defines a face of A.

1.1. **Proving** $dim A \geq n$.

Let $\mathscr{U}$ be a finite closed cover of A. We will show that if the diameter of each element of $\mathscr{U}$ is sufficiently small then the order of $\mathscr{U}$ is $n+1$.

(1) Let $F_0, \ldots, F_n$ be the $(n-1)$-dimensional faces of A with F_i the face opposite the vertex a_i. Explain why $\mathscr{C} = \{A \setminus F_i : i = 0, \ldots, n\}$ is an open cover of A.

(2) Let $\varepsilon > 0$ be the Lebesgue number of $\mathscr{C}$ and assume that each set in $\mathscr{U}$ has diameter less than ε. Explain why for each U_j there is some F_{n_j} which is disjoint from U_j.

(3) Explain why if v is a vertex of A, then there is some $U \in \mathscr{U}$ such that $v \in U$ and v is the unique vertex of A in U.

(4) Arbitrarily choose a function $b\colon \mathscr{U} \to \{F_k : k \in \{0, \ldots, n\}\}$ such that for each $U \in \mathscr{U}$, $U \cap b(U) = \varnothing$.. For $k \in \{0, \ldots, n\}$, let A_k be the union of the $U \in \mathscr{U}$ such that $b(U) = F_k$.

(5) We desire to show that there exists $x \in \bigcap_k A_k$. To do this, for a point $y \in A$, let $L(y)$ be the least k such that $y \in A_k$. Prove that if $\mathscr{T}$ is a triangulation of A, this labelling gives a Sperner labelling of the vertices of A.

(6) Let $\mathscr{T}^n$ be the n-th barycentric subdivision of $\mathscr{T}$. Explain why there is a completely labelled triangle $T_n \in \mathscr{T}^n$.

(7) Show that (perhaps after taking a subsequence) the vertices of the triangles T_n converge to a point $x \in A$.

(8) Prove that $x \in \bigcap_k A_k$.

(9) Explain why this implies that the order of $\mathscr{U}$ is at least $n+1$ and why $dimA \geq n$.

(10) We now show that for each $\varepsilon > 0$, there exists a finite closed cover $\mathscr{U}$ of A so that the order of $\mathscr{U}$ is $n+1$. Explain why this will complete the proof to show that $dimA = n$.

(11) For the $(m+1)$st barycentric subdivision of A, let $\mathscr{U}_m$ be the set having the stars of the vertices of the mth barycentric subdivision of A as elements. Sketch a picture in the case when the dimension of A is 2 and for when $m = 0$ and $m = 1$. Note that each $\mathscr{U}_m$ is a closed cover of A.

(12) Show that the order of $\mathscr{U}_m$ is $n+1$ and appeal to calculations from the text to show that the diameter of the sets in $\mathscr{U}_m$ goes to 0 as $m \to \infty$.

2. Simplicial approximations to homeomorphisms

As you know simplicial approximations are often not injective. This problem guides you through a proof that a simplicial approximation to a homeomorphism is, however, often surjective.

Let K and L be triangulations (i.e. simplicial complexes) such that $|K|$ and $|L|$ are homeomorphic surfaces. Let $g\colon |K| \to |L|$ be a homeomorphism. We know that after subdividing K enough times there is a simplicial approximation to h. So, without loss of generality, also assume that $f\colon K \to L$ is a simplicial approximation to g. We will show that f is surjective.

(1) Prove that for all $x \in |K|$, if $g(x)$ lies in the interior of a simplex $\sigma \in L$ then $f(x)$ lies in a face of σ (possibly σ itself).

    (Hint: Begin by reminding yourself that for each point in a simplicial complex, there is a unique simplex which contains that point in its interior. Prove the result by inducting on the dimension of the simplex in K containing x in its interior and think about what happens to the vertices of that simplex under both g and f.)

(2) We now prove that all the other simplices are in the range of f by inducting on the dimension.

    (a) Suppose that $e \in L$ is an edge with vertices v and w. Recall that g is onto. Using part (1), do a proof by contradiction to show that there is a point $x \in |K|$ with $g(x)$ and $f(x)$ both in the interior of e. Use the fact that f is simplicial to show that all of e is in the image of f.

    (b) Suppose now that $T \in L$ is a triangle not in the image of f. Let v_0, v_1, v_2 be its vertices. Let $A = \{x \in |K| : g(x) \in T\}$. Note that $g\colon A \to T$ is a homeomorphism (even though A need not be a simplex of K). For each point $x \in A$, by (1) we have that $y = f(x)$ lies in a vertex or edge of T. For $x \in A$, let $\tau_x \in K$ be the unique simplex such that x is in the interior of τ_x. Define

$$L(f(x)) = \min\{i : \exists z \in \sigma_x \text{ s.t. } f(z) = v_i\}$$

Prove that for any triangulation of T, this labelling gives a Sperner labelling on the vertices of the triangulation.

(c) Let T_m be a completely labelled triangle in the mth barycentric subdivision of T. After taking a subsequence, as $m \to \infty$, the vertices of T_m all converge to a point $y \in T$. We wish to show that y gives rise to a contradiction. Let $x = g^{-1}(y)$ and let τ_x be the simplex of K containing y in its interior.

(d) Case 1: τ_x is a triangle of K. In this case, take m large enough so that $T_m \subset g(\tau_x)$.

(e) Case 2: τ_x is an edge of K. In this case, take m large enough so that there are two triangles τ_1 and τ_2 having τ_x as a common edge and $T_m \subset g(\tau_1 \cup \tau_2)$ and the only edge of $g(\tau_1 \cup \tau_2)$ intersected by T_m is $g(\tau_x)$. Show that this leads to a contradiction.

(f) Case 3: τ_x is a vertex of K. In this case, derive a contradiction similarly to Case 2.

This problem has an obvious generalization to compact manifolds which are simplicial complexes of any dimension. Observe that this shows that the behaviour of maps between manifolds of the same dimension is quite different from the behaviour of maps from a lower-dimensional simplical complex into a higher dimensional simplicial complex.

Apart from the use of Sperner's Lemma, the argument of the preceding problem is typical when working with simplicial (or other) complexes – work inductively by induction across the dimension of the faces of the simplicial complex.