MA 331 HW 17: Taking the path less travelled

1. Reading

Read Sections II.1 of Edelsbrunner and Harer. Pay particular attention to:

- the definition of orientable vs. non-orientable surface

2. Problems

(1) Do problem 2 on page 24 of Edelsbrunner-Harer.

(2) (*) Let $I = [0,1]$. Let X be a path connected topological space and let $* \in X$. Let $P(X,*)$ be the set of all paths

$$\{\gamma\colon I \to X : \gamma(0) = \gamma(1) = *\}$$

For $\gamma, \alpha \in P(X,*)$, define $\gamma \sim \alpha$ if and only if there is a homotopy $F\colon I \times I \to X$ with the following properties:

- For all $t \in I$, $F(t,0) = \gamma(t)$ and $F(t,1) = \alpha(t)$.
- For all $s \in I$, $F(0,s) = F(1,s) = *$.

We can concatenate paths as follows. Let $\gamma, \gamma' \in P(X,*)$. Define

$$\gamma' \cdot \gamma(t) = \begin{cases} \gamma(2t) & t \in [0,1/2] \\ \gamma'(2t-1) & t \in [1/2,1] \end{cases}$$

Prove the following:

(a) $\sim$ is an equivalence relation on $P(X,*)$

(b) If $\alpha \sim \beta$ and $\gamma \sim \phi$, then $\gamma \cdot \alpha \sim \phi \cdot \beta$.

(3) Prove that a contractible space is path-connnected. (Hint: the homotopy realizing the contraction, for a fixed point in the space, will trace out a path.)

(4) Suppose that $\gamma\colon S^1 \to S^n$ (for $n \geq 2$) is continuous and that there is a point $N \notin \gamma(S^1)$. Prove that there is a homotopy from γ to a constant map. (i.e. the path γ is contractible.)