MA 331 HW 16: What are we covering today?

1. Reading

Read Sections I.1 and I.2 of Edelsbrunner and Harer. Pay particular attention to the algorithms for Disjoint Set Systems and for determining if a point is inside or outside a polygon. Also observe how topics we've spent a lot of time on are dealt with succinctly. This should help you calibrate your future reading expectations.

2. Problems

(1) Prove that if X and Y are compact topological spaces with $Y \subset X$ then the topological dimension of Y is at most that of X.

(2) (*) Prove that the topological dimension of a finite (hence, compact) graph is 0 if the graph has no edges and is 1 if the graph has at least one edge.

(3) A topological space X is **locally compact** if for every $x \in X$ there is a compact neighborhood of x. (Recall that a neighborhood of x is any set which contains an open set containing x.)

Suppose that X is a locally compact, Hausdorff topological space. Let ∞ be a point not in X and define $\widehat{X} = X \cup \{\infty\}$. Define a topology on $\widehat{X}$ as follows. A set $U \subset \widehat{X}$ is open if one of the following holds:

- $\infty \notin U$ and U is open in X.
- There is a compact subset $C \subset X$ such that $U = \widehat{X} - C$.

Prove that $\widehat{X}$ is compact and Hausdorff. Also show that it is the unique topology on the set $\widehat{X}$ which makes $\widehat{X}$ compact and Hausdorff and for which the given topology on $X \subset \widehat{X}$ is equal to the subspace topology.

(The space $\widehat{X}$ is the "one-point" compactification of X. You may take it for granted that the topology on $\widehat{X}$ really is a topology.)

(4) For each $n \in \mathbb{N}$, let $R_n = \mathbb{R}$. Define $R = \sqcup_n R_n$ (i.e. R is the disjoint union of countably many copies of $\mathbb{R}$). Give as rigorous a proof as you can (which may fall short of complete rigor) that the one point compactification of R is homeomorphic to the Hawaiian Earring Space (i.e. the union of circles with center $(0, 1/n)$ and radius $1/n$ for all $n \in \mathbb{N}$.)

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(5) Prove that if a topological graph G is path connected then it has the property that for all $a, b \in G$ there exists a sequence of edges $e_1, \ldots, e_n$ so that $a \in e_1$ and $b \in e_n$ and for all i, the edges e_i and e_{i+1} share a vertex.

(Hint: Take a path from a to b which minimizes the number of vertices it intersects. If it intersects some vertex more than once, you can "shorten" the path by cutting out all the intermediate stuff. Since $[0,1]$ is compact, the path cannot fully transverse infinitely many edges. Your task is to make all this as precise, but understandable, as you can.)

(6) The one-point compactification of $\mathbb{R}$ is S^1. Is there a general way of compactifying so that we add a point for each "end" of the space, rather than just a single point? There is – it's called the endpoint compactification. The purpose of this problem is to think about how one might set about constructing such an object.

Suppose that X is a connected topological space with the property that there is a sequence $K_n \subset X$ of compact sets such that $X = \bigcup_{n \in \mathbb{N}} K_n$ and, for all n, $K_n \subset K_{n+1}$. Say that a sequence (a_k) is **exiting** if for each n, only finitely many terms of (a_k) lie in K_n. Define an equivalence relation $\sim$ on the set of exiting sequences by declaring that $(a_k) \sim (b_p)$ if for each n, all but finitely many terms of (a_k) and (b_p) lie in the same connected component of $X - K_n$. Speculate about how the equivalence classes of exiting sequences might be used to compactify X. You should say what the new set will be and how the topology should be defined. You should also carefully list all the facts that one might like to prove to verify that you have a useful construction. *You do not need to give any rigorous proofs in this problem – just conduct a careful thought experiment!*