MA 331 HW 15: Is the Mayflower Compact?

1. Definitions

If X is a topological space, an **open cover** of X is a collection $\mathscr{U}$ of open sets in X such that for every $x \in X$, there exists $U \in \mathscr{U}$ with $x \in U$. The space X is **compact** if for every open cover $\mathscr{U}$ of X there exists a finite subset $\mathscr{U}' \subset \mathscr{U}$ which is also a cover of X. (We call $\mathscr{U}'$ a **finite subcover**.) A subset K of a topological space X is **compact** if with the subspace topology K is compact.

2. Reading

Study the proofs of Theorems 2.9, 2.11 and 2.12 from Section 5.2 of Mendelson.

3. Problems

(1) (*) Suppose that X is compact and that $f\colon X \to Y$ is a continuous surjection. Prove that Y is compact.

(2) (*) Suppose that X is a compact topological space and that for each $n \in \mathbb{N}$, $K_n \subset X$ is a closed, non-empty subset. Suppose also that for all n, $K_{n+1} \subset K_n$. Prove that $\bigcap_n K_n \neq \varnothing$.

(3) (*) Prove that a topological graph G is compact if and only if G has finitely many edges and vertices.

(4) (Challenging!) Suppose that X and Y are topological spaces. Let $C(X,Y)$ be the set of continuous functions $X \to Y$. We give $C(X,Y)$ a topology $\mathscr{T}$ (called the **compact-open topology**) as follows: Let $\mathscr{U}$ be the set of all subsets $U \subset C(X,Y)$ such that for each $U \in \mathscr{U}$ there exists a compact set $K \subset X$ and an open set $V \subset Y$ such that $f \in U$ if and only if $f(K) \subset U$. We let $\mathscr{T}$ be the smallest topology on $C(X,Y)$ such that $\mathscr{U} \subset \mathscr{T}$. (That is, $\mathscr{U}$ is a sub-base for $\mathscr{T}$). Give an example of spaces X and Y for which you can completely determine $C(X,Y)$.