MA 331 HW 14: Connected and Path Connected

1. Definitions

A **topological space** X is **path-connected** if for every $x, y \in X$ there exists a map $f\colon [0,1] \to X$ so that $f(0) = a$ and $f(1) = b$.

Suppose that X is the disjoint union of copies of the interval $I = [0,1]$. The **vertices** of X are the copies of $\{0\}$ and $\{1\}$ in each copy of the interval. The edges of X are the copies of the intervals $[0,1]$. Let $\sim$ be an equivalence relation on X such that if $x \sim y$ and $x \neq y$ then both x and y are vertices of X. By definition the quotient space $X/\sim$ is a (topological) **graph**. The images (under the quotient map) of the vertices of X are called the **vertices** of $X/\sim$. The images of the edges of X are called the **edges** of $X/\sim$.

2. Problems

(1) (*) Suppose that $X\colon Y$ is a surjective (continuous) map between topological spaces X and Y. Assume that X is path connected, prove that Y is path connected.

(2) (*) Suppose that X is a path-connected topological space. Prove that X is connected.

(3) (*) Suppose that for each $\alpha \in \Lambda$, X_α is a topological space. Let $X = \prod_{\alpha \in \Lambda} X_\alpha$ with the product topology.

   (a) If each X_α is connected, prove that X is also connected. (See Theorem 2.12, page 118 of Mendelson)

   (b) If each X_α is path connected, prove that X is also path connected.

(4) (Challenging!) Let C be the union of the circles C_n of radius $1/n$ and center $(0, 1/n)$ in $\mathbb{R}^2$. Let $p \in C$ be the point where they are all tangent. Prove that there is a path $\gamma\colon [0,1] \to C$ so that $\gamma(0) = \gamma(1) = p$ and the image of γ is all of C.

(5) (Challenging!) *Attempt* to prove that a topological graph G is path connected if and only if for every $a, b \in G$, there is a (finite) sequence of edges $e_1, \ldots, e_n$ such that $a \in e_1$, $b \in e_n$ and for all $i \in \{1, \ldots, n-1\}$, the edges e_i and e_{i+1} share a vertex. (Note we are not assuming that G has only finitely many edges.)

If you are unable to prove the theorem, articulate what the difficulties are and what you would need to know to get past it.