MA 331 HW 13: Feel connected!

1. Reading

(1) Reread the Abrams-Ghrist article and take the time to understand more of it.

2. Definitions

A **topological space** X is **connected** if the only subsets of X which are both open and closed are X and $\varnothing$. Equivalently, X is **disconnected** if there exist disjoint non-empty open sets P and Q such that $X = P \cup Q$. If X is a topological space and if $A \subset X$, we say that A is **connected** if it is connected as a topological space with the subspace topology. A connected nonempty subset of X which is both open and closed is called a **connected component** of X. (We also say that $\varnothing$ is a connected component of $\varnothing$.)

3. Problems

(1) (*) Suppose that $X\colon\ Y$ is a surjective (continuous) map between topological spaces X and Y. Suppose that Y is disconnected. Prove that X is also disconnected.

(2) (*) Suppose that X is a connected topological space and that $\sim$ is an equivalence relation on X. Prove that the quotient space $X/\sim$ is connected.

(3) Prove that if X and Y are homeomorphic then they have the same number of connected components. (That is, there is a bijection between their connected components.)

(4) Give an example of topological spaces X and Y and a continuous function $f\colon\ X \to Y$ such that X is disconnected and Y is connected.

(5) Look back through some (say 3) of the stranger examples of metric spaces and topological spaces that we've seen and determine if they are connected or disconnected. You should give compelling arguments, but not necessarily totally rigorous proofs.