MA 331 HW 11: What are you gluing?

1. Review

Suppose that X and Y are topological spaces and that $f\colon A \to B$ is continuous where $A \subset X$ is closed and $B \subset Y$. Recall the following :

- $X \sqcup Y$ is the disjoint union of X and Y – this is the union of X and Y if they are disjoint and the union of (say) $X \times \{1\}$ and $Y \times \{2\}$ if they are not disjoint (note that those two sets are disjoint). Its topology is just the union of the topologies on X and Y.
- We can define an equivalence relation on $X \sqcup Y$ by demanding that for all $a \in A$, $a \sim f(a)$ and then demanding that $\sim$ be reflexive, symmetric, and transitive. In other words $\sim$ is the equivalence relation:
  - for all $z \in X \sqcup Y$, $z \sim z$
  - for all $a, a' \in A$ with $f(a) = f(a')$ we have $a \sim a'$
  - for all $b \in B$ and $a \in A$ with $f(a) = b$ we have $a \sim b$ and $b \sim a$.
- The quotient space $X \cup_f Y$ is $(X \sqcup Y)/\sim$ with the quotient topology.

2. Problems

(1) The point of the next problem is a thought experiment - it doesn't require you to prove anything.

As described in the Mendelson reading from the last handout, we can also define a quotient (or identification) topology as follows:

If $q\colon X \to Y$ is a surjection with X a topological space, the **quotient topology** on Y is the topology $\{U \subset Y : q^{-1}(U) \text{ is open in } X\}$.

Consider the following two maps:

$$\begin{aligned} q_1 &: \quad \mathbb{R} \to S^1 \\ q_2 &: \quad \mathbb{R} \to \mathbb{R}/\sim \end{aligned}$$

where $q_1(t) = (\cos 2\pi t, \sin 2\pi t)$ and $q_2(t) = [t]$ where $\sim$ is the equivalence relation on $\mathbb{R}$ defined by $x \sim x + n$ for all $n \in \mathbb{Z}$. Explain why the q_1 and q_2 are in some sense "the same" quotient map, even thought technically their codomains are different. In our explanation, be sure to specify what it is that makes the codomains different.

(2) This result is called the gluing lemma and is quite handy.

Suppose that X and Y are topological spaces and that $A \subset X$ is closed and $B \subset Y$. Let $f\colon A \to B$ be continuous and let $Z = X \cup_f Y$. Suppose that W is yet another topological space and that $g_X\colon X \to W$ and $g_Y\colon Y \to W$ are continuous functions. Suppose also that whenever $a \in A$ and $b \in B$ with $f(a) = b$ we also have $g_X(a) = g_Y(b)$. We can define a function

$$h\colon Z \to W$$

defined by

$$h(z) = \begin{cases} g_X(x) & \text{if } z = [x] \text{ with } x \in X \\ g_Y(y) & \text{if } z = [y] \text{ with } y \in Y \end{cases}$$

Prove that h is well-defined and that it is continuous.

Hint for proving it is continuous: Let $q\colon X \sqcup Y \to Z$ be the quotient map. Figure out what kind of open sets in $X \sqcup Y$ get mapped to open sets in Z. More hints available on request.

(3) For all α in some index set Λ, let X_α be a topological space and let $*_\alpha \in X_\alpha$. Define the **one-point union** (or **wedge product**) of the X_α along the $*_\alpha$ to be the space $(\sqcup_\alpha X_\alpha)/\bigcup_\alpha *_\alpha$. (Here $\sqcup$ means "disjoint union". If the sets aren't disjoint, replace them by copies that are.) In otherwords, we take all the X_α and then crush the separate points $*_a$ down to a single point.

(a) Suppose that $X_1 = S^1$ and that $X_2 = S^1$. Let $a_1, b_1 \in X_1$ and $a_2, b_2 \in X_2$. Prove that the one-point union of X_1 and X_2 along a_1 and a_2 is homeomorphic to the one-point union of X_1 and X_2 along b_1 and b_2.

(b) For each $n \in \mathbb{N}$ let $X_n = S^1$ and choose a point $*_n \in X_n$. Let X be the one point union of the X_n along the $*_n$. Let C_n be the circle in $\mathbb{R}^2$ of radius $1/n$ and center $(0, 1/n)$. Let $C = \cup_n C_n \subset \mathbb{R}^2$ with the subspace topology. Try to prove that X is not homeomorphic to C.