MA 331 HW 10: Odds and Ends

1. Definitions

A topological space X is **sequentially compact** if every sequence in X has a convergent subsequence. (Note: we are not assuming X is a metric space!) A function $f\colon X \to Y$ is **sequentially continuous** if whenever a sequence (x_n) in X converges to a point $x \in X$ then the sequence $(f(x_n))$ in Y converges to the point $f(x) \in Y$.

2. Reading

(1) Read Section 1.8 (Nowadays we say "quotient topology" instead of "identification topology")

3. Problems

(1) (*) Do problems 2 and 4 on page 106 of Mendelson. Be prepared to present them to a classmate. **MISTAKE in 2**

(2) Suppose that $f\colon X \to Y$ is sequentially continuous and that X is sequentially compact. Prove that the image $f(X) \subset Y$ is sequentially compact when given the subspace topology from Y.

(3) Let $f\colon X \to Y$ be a function and X, Y Hausdorff topological spaces. Prove that f is sequentially continuous if it is continuous.

(4) (Bonus Challenge!) Find examples of spaces X, Y and a function $f\colon X \to Y$ where f is sequentially continuous, but not continuous.

(5) Suppose that $f\colon X \to Y$ is a function between topological spaces. Let $\{U_\alpha\}$ be a basis for the topology on Y. Suppose f has the property that for each basis element U_α, the set $f^{-1}(U_\alpha)$ is open in X. Prove that X is continuous.

(6) A T_1 topological space X is **regular** if for any point $x \in X$ and any closed set $F \subset X$ with $x \notin F$ there exist disjoint open sets U and V with $x \in U$ and $F \subset V$. (In other words, points can be separated from closed sets by open sets.)

   (a) Give an example of a regular space.

   (b) Prove that regular spaces are Hausdorff but not all Hausdorff spaces are regular.

(c) Prove that if X is regular and if $A \subset X$ is closed, then the quotient space X/A is Hausdorff.

(7) The set $M_n(\mathbb{R})$ of $n \times n$ matrices of real numbers can be identified with $\mathbb{R}^{n^2}$ (there are several natural ways of doing this – just pick one). Give $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ the product topology (perhaps by using the max-metric on it). We consider several special subsets of M_n:

- The set $GL_n(\mathbb{R})$ of invertible $n \times n$ matrices
- The set $SL_n(\mathbb{R})$ of $n \times n$ matrices with determinant equal to 1.

ADD HINT ABOUT DETERMINANT

Prove that $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$ and that $SL_n(\mathbb{R})$ is a closed subset of both $M_n(\mathbb{R})$ and $GL_n(\mathbb{R})$. Is either of these sets sequentially compact?