MA 331 Exam 2 Study Guide

1. Definitions

Know the following definitions:

(1) Connected space
(2) Path-connected space
(3) Connected component
(4) Compact
(5) one-point compactification
(6) Sequentially compact
(7) Simplex
(8) Simplicial complex
(9) Star Condition
(10) Simplicial Approximation
(11) Simplicial Map
(12) Simply Connected space
(13) order of a closed finite covering of a compact metric space
(14) topological dimension of a compact metric space
(15) Lebesgue number of an open covering of a metric space
(16) Homotopy of maps
(17) Homotopy equivalent spaces
(18) contractible space

2. Little Theorems

Be able to give proofs of the following results. On the actual exam you will be given 4 of the following and asked to prove 2.

(1) A closed subset of a compact space is compact
(2) A compact subset of a Hausdorff space is closed
(3) If $F_1 \supset F_2 \supset F_3 \supset \dots$ is a sequence of non-empty closed subsets of a compact space X then $\bigcap_k F_k \neq \varnothing$.
(4) A compact metric space is sequentially compact
(5) Compactness and connectedness are topological properties
(6) The one-point compactification of a locally compact, Hausdorff space is compact and Hausdorff
(7) A space is connected if and only if there is no continuous surjection to $\{1,2\}$ (with the discrete topology)
(8) The circle S^1 and the interval $[0,1]$ are not homeomorphic

(9) The Lebesgue number of an open covering of a compact subset of $\mathbb{R}^n$ is positive (i.e. the Lebesgue Lemma)

(10) The interval $[0,1]$ is connected

(11) The topologists' sine curve is connected but not path connected.

(12) A graph is compact if and only if it has finitely many edges and vertices

(13) A graph is connected if and only if for any two vertices a and b there is a finite sequence of edges $e_1, \dots, e_n$ such that $a \in e_1$ and $b \in e_n$ and e_i and e_{i+1} share a vertex.

(14) If $f\colon X \to Y$ is a continuous bijection and if X is compact and if Y is Hausdorff then f is a homeomorphism

(15) If $f\colon X \to Y$ is a continuous surjection and if X is compact, then so is Y.

(16) The Hawaiian Earring Space and the one-point union of countable many circles are not homeomorphic.

(17) Contractible spaces are both path-connected and simply connected.

(18) $\mathbb{R}^n$ is contractible

(19) The sphere S^n for $n \geq 2$ is simply-connected.

(20) Topological dimension for compact metric spaces is a homeomorphism invariant

(21) If $Y \subset X$ are compact metric spaces then the topological dimension of Y is at most that of X.

(22) Two finite graphs are homotopy equivalent if and only if they have the same euler characteristic.

3. Big Theorems

Be able to give moderately detailed proofs of the following. On the actual exam you will be given 4 and asked to provide a moderately detailed outline for 2.

(1) Sperner's Lemma
(2) Brouwer Fixed Point Theorem (assuming Sperner's Lemma)
(3) The Jordan Curve theorem for polygonal curves
(4) The property of maps $X \to Y$ being homotopic is an equivalence relation
(5) The extreme value theorem for compact sets
(6) If $U \subset \mathbb{R}^n$ is an open set homeomorphic to an open set $V \subset \mathbb{R}^m$ then $n = m$ (Invariance of Dimension) Note: You do not need to know how to prove that the topological dimension of an n-simplex is n, but you may use that fact.

(7) If G' is a graph obtained by collapsing an edge e of a graph G having distinct endpoints, then G and G' are homotopy equivalent

(8) If K and L are simplicial complexes and if $g\colon\ |K| \to |L|$ is continuous and satisfies the star condition then there is a simplicial approximation $f\colon\ K \to L$ to g.

(9) If K and L are simplicial complexes and if $g\colon\ |K| \to |L|$ is continuous, then there is an n such that if K^n is the nth barycentric subdivision of K, then there is a simplicial approximation $f\colon\ K^n \to L$ to g.

(10) Every homeomorphism $S^1 \to S^1$ is homotopic to either the identity map or to the result of reflecting the circle across a line passing through the origin. (This is a fairly challenging, new problem. Try using simplicial approximation, but note that a simplicial approximation to a homeomorphism need to be a homeomorphism. After applying Simplicial approximation, to make the map "nice", make careful redefinitions to turn it back into a homeomorphism. This is a lot like the problem where you show that a graph is path connected if and only if there is a path of edges between any two points)