MA 274: Induction Examples

1. Plain Old Mathematical Induction

Theorem (Mathematical Induction)**.** Suppose that for each $n \in \mathbb{N}$, $P(n)$ is a statement. Suppose also that:

- $P(1)$ is true.
- If $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $A = \{n \in \mathbb{N} : P(n) \text{ is true.}\}$. By hypothesis,

- $1 \in A$
- If $n \in A$, then $n+1 \in A$.

Hence, by the 5th Peano Axiom for the Natural Numbers, $A = \mathbb{N}$. That is, $P(n)$ is true, for all $n \in \mathbb{N}$. □

Theorem 1.1. If $n \in \mathbb{N}$, then for some $k \in \mathbb{N} \cup \{0\}$, n is equal to one of $3k$, $3k+1$, or $3k+2$.

Proof. We prove this by induction.

Base Case: If $n = 1$, then $n = 3(0)+1$.

Inductive Step: Assume that there exists $k \in \mathbb{N} \cup \{0\}$ such that n equals one of $3k$, $3k+1$, or $3k+2$. We will show that there exists $m \in \mathbb{N} \cup \{0\}$ so that $n+1$ equals one of $3m$, $3m+1$, or $3m+2$.

Case a: Suppose that $n = 3k$.

In this case $n = 3k+1$, so letting $m = k$ gives us our result.

Case b: Suppose that $n = 3k+1$.

In this case, $n+1 = (3k+1)+1 = 3k+2$. Once again, we have our result.

Case c: Suppose that $n = 3k+2$.

In this case, $n+1 = (3k+2)+1$ and so $n = 3k+3 = 3(k+1)$. Let $m = k+1$. Then $n+1 = 3m$ and we have our result. □

Theorem 1.2. Suppose that G_n is the result of removing a single square from a $2^n \times 2^n$ grid of square tiles. Then G_n can be tiled by L-shapes made up of 3 square tiles.

Proof. We induct on n.

Base Case: If $n = 1$, Consider a 4×4 grid. Removing any of the four squares results in an L-shape made up of 3 square times. So G_1 can be tiled by the L-shapes.

Inductive Step: Assume that G_n can be tiled by L-shapes made up of 3 square tiles. We will prove that G_{n+1} can also be tiled in such a way.

Let S_n and S_{n+1} be the $2^n \times 2^n$ grid and the $2^{n+1} \times 2^{n+1}$ grid, respectively. Since $2^{n+1} \times 2^{n+1} = (2 \cdot 2^n) \times (2 \cdot 2^n)$, S_{n+1} can be divided into four copies of S_n all sharing the middle vertex of G_{n+1}. Call these grids T_1, T_2, T_3, and T_4. G_{n+1} is made by removing a square σ from S_{n+1}. The square σ must lie in exactly one of $T_1, \ldots, T_4$. Without loss of generality, we may assume that $\sigma \subset T_1$. Then $T_1 - \sigma$ can be tiled by the L shapes by the induction hypothesis. Place an L shape so that one square of the shape lies in each of T_2, T_3, and T_4. (The inside corner of the shape is at the center point of S_{n+1}.) Removing that L-shape from $T_1 \cup T_2 \cup T_3$ results in 3 copies of G_n. By the induction hypothesis, the rest of T_1, T_2, and T_3 can be tiled by the L-shapes. We have, therefore, tiled all of G_{n+1} by L-shapes. □

2. Complete Induction

Theorem (Complete Induction)**.** Suppose that for each $n \in \mathbb{N}$, $P(n)$ is a statement. Suppose also that:

- $P(1)$ is true.
- If $P(k)$ is true for all $k \leq n$, then $P(n+1)$ is true.

Then for all $n \in \mathbb{N}$, $P(n)$ is true.

Proof. Let $Q(n)$ be the statement that $P(k)$ is true for all $k \leq n$. We will prove that $Q(n)$ is true for all $n \in \mathbb{N}$ by induction.

Base Case: Since $Q(1) = P(1)$, $Q(1)$ is true by hypothesis.

Inductive Step: We assume that $Q(n)$ is true and we will prove that $Q(n+1)$ is true. Since $Q(n)$ is true, by the definition of $Q(n)$, $P(k)$ is true for all $k \leq n$. By hypothesis, this implies that $P(n+1)$ is true. Thus, $Q(n+1)$ is true, since $Q(n+1) = Q(n) and P(n+1)$.

Hence, by induction $Q(n)$ is true for all n. Since the truth of $Q(n)$ implies the truth of $P(n)$, $P(n)$ is also true for all $n \in \mathbb{N}$. $\square$

3. Well Ordering Principle

Theorem. Suppose that $A \subset \mathbb{N}$ is non-empty. Then A has a least element.

Proof. We prove the contrapositive. Let $A \subset \mathbb{N}$ be a set without a least element. We prove that $A^c = \mathbb{N}$ (in other words, $A = \varnothing$) by complete induction.

Base Case: Since $1 \in \mathbb{N}$ is the smallest integer, if $1 \in A$ then A would have a least element, contradicting our hypothesis. Hence, $1 \in A^c$.

Inductive Step: We assume that $k \in A^c$ for all $k \leq n$ implies that $(n+1) \in A^c$.

Suppose that $k \in A^c$ for all $k \leq n$. Then, if $(n+1) \in A$ the number $(n+1)$ would be the least element in A, since every number in $\mathbb{N}$ smaller than $(n+1)$ lies in A^c. Hence, $(n+1) \in A^c$ as well.

Since we have proved both the base case and the inductive step, by the principle of complete induction $n \in A^c$ for all $n \in \mathbb{N}$. Hence, $A^c = \mathbb{N}$ and $A = \varnothing$. $\square$

Here is the last piece we needed for our proof that $\sqrt{2}$ is irrational.

Theorem 3.1. Every positive rational number can be expressed as $\frac{a}{b}$ with $a, b \in \mathbb{N}$ and where the only common positive divisors of a and b is 1.

Proof. Let ρ be a positive rational number. Let $A = \{a \in \mathbb{N} : \exists b \in \mathbb{N} \text{ with } \rho = a/b\}$. Since ρ is a positive rational number, $A \neq \varnothing$. By the Well Ordering Principle, A has a least element x. Since $x \in A$, there exists $y \in \mathbb{N}$ such that $\rho = x/y$. We claim that x and y have no common divisor in $\mathbb{N}$ except 1.

Suppose that x and y are both multiples of $r \in \mathbb{N}$. Thus, there exists $x', y' \in \mathbb{N}$ such that $x = rx'$ and $y = ry'$. By the basic properties of arithmetic $x' \leq x$. Then,

$$r = \frac{x}{y} = \frac{rx'}{ry'} = \frac{x'}{y'}.$$

Thus, $x' \in A$. Since x is the least element of A, $x \leq x'$. We also have $x' \leq x$ and so $x = x'$. This implies $r = 1$, as desired. $\square$