## MA 274: Peano's Axioms

Pretend that you know nothing except for the rules of logic (as expressed in the text) and the idea of "set", "subset", and "element of a set".

Peano's axioms for a set $N$ are:

(P1) 1 is an element of the set $N$.

(P2) For each element $n$ of $N$, there exists an element $S(n)$ in $N$. ($S(n)$ is called the successor of $n$.)

(P3) For all $n$ in $N$, $S(n) \neq 1$.

(P4) For all $m$ in $N$ and for all $n$ in $N$, if $m \neq n$, then $S(m) \neq S(n)$

(P5) If $A$ is a subset of $N$ such that 1 is an element of $A$ and if for all $x$ in $A$, $S(x)$ is an element of $A$, then $A = N$.

**Problems:**

(1) (G) Show that the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ satisfies (P1) - (P4). Part of your explanation should be specifying what $S(n)$ is for each $n \in \mathbb{N}$. In showing that $\mathbb{N}$ satisfies (P1) - (P4), you should give as rigorous a proof as you can.

(2) (G) Do you think $\mathbb{N}$ also satisfies (P5) - why or why not? You do not need to give a proof of your answer.

(3) (G) Give an example of a set $N$ and a function $S$ such that $N$ and $S$ satisfy (P1) - (P3), but do **not** satisfy (P4). Does your example satisfy (P5)? Why or why not?

(To answer this last question, you are allowed to come up with whatever set $N$ and whatever function $S$ you want. I suggest though that you not try to be too adventurous. This exercise shows that it is impossible to prove the statement (P4) simply from statements (P1), (P2), and (P3). In other words, (P4) is *independent* of (P1), (P2), and (P3).)