MA 274: Traditional Axioms for Euclidean Geometry

Common Notions (Rules of Logic)

(1) Things which equal the same thing also equal each other.

(2) If equals are added to equals, then the wholes are equal.

(3) If equals are subtracted from equals, then the remainders are equal

(4) Things which coincide with one another equal one another

(5) The whole is greater than the part.

Postulates

(1) For every two distinct points, there exists a unique line that passes through them.

(2) Given a finite straight line, it is possible to continue it, as a straight line.

(3) Given a point and a radius, there exists a circle with that center and radius

(4) All right angles equal one another.

(5) (Playfair's version) Given a line and a point not on the line, there exists a unique line through the given point parallel to the given line.

Selected Definitions

(1) A point is that which has no part

(2) A line is a breadthless length

(4) A straight line is a line which lies evenly with the points on itself

(15) A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

(23) Two lines are parallel if and only if when extended indefinitely they never have any points in common.

Two Selected Propositions

Theorem (Proposition 17)**.** In any triangle, the sum of any two angles is less than the sum of two right angles.

Proof. Let ABC be a triangle and let ABC and BCA be two of its angles. Extend the line BC to a line BD so that C is between B and D. (Postulate 2) The angle ACD is an exterior angle of the triangle ABC. By Proposition 16, it is greater than the interior opposite angle ABC. Adding the angle ACB to the angles ACD and ABC, shows that

$$ACD + ACB > ABC + ACB.$$

The sum $ACD + ACB$ is equal to the sum of two right angles (Proposition 13). Thus, the sum of the angles ABC and BCA is less than two right angles. □

Theorem (Proposition 32)**.** In any triangle, the sum of the interior angles of the triangle equals the sum of two right angles.

Proof. Let ABC be a triangle and extend the side BC to D. (Postulate 2). Draw a line CE through C parallel to AB. (Postulate 5) Since AB is parallel to CE and since AC falls on them, the alternate angles BAC and ACE equal one another (Proposition 29). Since BD falls on AB and CE, the exterior angle ECD equals the interior and opposite angle ABC (Proposition 29). The angle ACE equals the angle BAC; thus,

$$ACD = BAC + ABC.$$

Add ACB to each. Then:

$$ACD + ACB = BAC + ABC + ACB.$$

Since $ACD + ACB$ is equal to the sum of two right angles, the sum of the interior angles of ABC is equal to two right triangles. □