MA 253 Homework Problems 3

Homework has both a "Reading" portion and a "Problems" portion. It is essential that you do the reading by the next class. The reading assignments are posted on a separate webpage. Don't forget to do them!

The following problems are due on **Wed. September 25**. All page numbers and section numbers refer to the 5th edition of Bretscher's Linear Algebra. Note that most odd numbered problems have solutions in the back of the text. Problems without solutions are worth more points than those with solutions.

(1) Section 2.3 (page 85)

  (a) Problems 15, 16 (These concern "block matrices". We won't discuss these in class, but they are occassionally useful.)
  (b) Problem 30. (This is an important result from 2-dimensional geometry)
  (c) Problems 69, 70. (These problems concern an important use of linear algebra in the internet. Don't forget to use *Mathematica* to simplify calculations!)
  (d) Problem 84. (Hint: Recall that you can find such an X by augmenting A with the identity matrix and row reducing. What does the rank of A tell you about what you end up with?)

(2) Section 2.4 (page 71)

  (a) Problems 3, 4, 5, 8, 12, 20, 28. You may use *Mathematica* to do row reductions, but you may not use its ability to find inverse matrices. The point is to understand the process of how inverse matrices are found.
  (b) Problem 30. Be sure to show and explain your work!
  (c) Problem 41. You can (and should) do this without writing down any matrices!
  (d) Problems 42, 83.
  (e) Problem 45. The results of this problem are important in applied mathematics. It explains why in applied mathematics, we (almost) never invert a matrix.

(f) Problems 67 - 75. Read the instructions for these problems! If the statement is true explain why. If the statement is false, give a counter-example.

(3) For a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (not necessarily linear) we can write:

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

where f_1 and f_2 are functions $\mathbb{R}^2 \to \mathbb{R}$. At a point $(a,b) \in \mathbb{R}^2$ we can consider the **derivative** of f:

$$Df(a,b) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) \\ \frac{\partial f_2}{\partial x}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \end{pmatrix}$$

If you haven't seen it before, $\frac{\partial f_1}{\partial x} f_1(a,b)$ means: take the derivative of f_1, thinking of x as the variable and then plug in (a,b). For example, if $f(x,y) = (x^2 + y^3, e^x + e^{y^2})$ we have:

$$Df(x,y) = \begin{pmatrix} 2x & 3y^2 \\ e^x & 2ye^{y^2} \end{pmatrix}.$$

The function f is **smooth** if $Df(a,b)$ exists and has non-zero determinant for every point (a,b).

(a) Is the example given above smooth? Why or why not?

(b) Suppose that the determinant of $Df(a,b)$ is zero at a point (a,b). Using your knowledge of what the determinant measures geometrically, give an informal description of what is happening at the point (a,b) – in particular why might we call f "non-smooth" at that point?

(c) Suppose, now, that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function. Explain why the determinant of $Df(a,b)$ is either always positive or is always negative. If it is always positive, we say f is "orientation-preserving" and if it is negative, we say it is "orientation reversing". Using your knowledge of the derivative, explain why this terminology makes sense.

(d) Show that the function $f(x,y) = (e^x, e^y)$ is smooth and orientation-preserving.

(e) Show that the function $f(x,y) = (-e^x, e^y)$ is smooth and orientation-reversing.

(This problem is intended to give you just a taste of how techniques from linear algebra can be used to study non-linear functions.)