MA 253 Homework Problems 12

Homework has both a "Reading" portion and a "Problems" portion. It is essential that you do the reading by the next class. The reading assignments are posted on a separate webpage. Don't forget to do them!

Also: Don't forget to work on your projects!

All page numbers and section numbers refer to the 5th edition of Bretscher's Linear Algebra. Note that most odd numbered problems have solutions in the back of the text. Problems without solutions are worth more points than those with solutions.

(1) Suppose that X is a vector space with basis $\{v_1, \ldots, v_n\}$ and that Y is a vector space. Explain how we can define a linear map $T\colon X \to Y$ simply by defining $T(v_1), T(v_2), \ldots, T(v_n)$. (That is, given $T(v_1), \ldots, T(v_n)$ you must explain how to define $T(x)$ for any $x \in X$.)

(2) Prove that the vector spaces M_n (of $n \times n$ matrices) and R^{n^2} are isomorphic.

(3) Let P_4 be the polynomials of degree at most 4 and P_5 the polynomials of degree at most 5. Define a function $T\colon P_4 \to P_5$ by

$$T(p(x)) = \int_0^x p(x)\,dx$$

(a) Show that T is a linear transformation
(b) Find basis for the kernel of T.
(c) Find a basis for the image of T.

(4) Let $\mathscr{S}$ be the set of sequences of real numbers. Let $s\colon S \to S$ be the shift transformation:

$$s(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$$

(a) Find an eigenvector for s
(b) Find an eigenvector for $s^2 = s \circ s$ which is not an eigenvector for s.

(5) Let $SF = \{f\colon \mathbb{R}^3 \to \mathbb{R} : f \text{ is infinitely differentiable.}\}$ and let $VF = \{F\colon \mathbb{R}^3 \to \mathbb{R}^3 : F \text{ is infinitely differentiable}\}$. Notice that if $f \in SF$ then all partial derivatives of f are defined and notice that if $F \in VF$, then

$$F(x,y,z) = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}$$

where F_1 and F_2 are functions such that all partial derivatives of F_1 and F_2 and F_3 are defined.

(The set SF is the set of *scalar fields* in $\mathbb{R}^3$ and the set VF is the set of *vector fields* in $\mathbb{R}^3$.)

(a) Explain why SF and VF are linear spaces.

(b) Consider the function (called the *gradient*) $\nabla\colon SF \to VF$ defined by

$$\nabla f(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x} f(x,y,z) \\ \frac{\partial}{\partial y} f(x,y,z) \\ \frac{\partial}{\partial z} f(x,y,z) \end{pmatrix}$$

Explain why the function ∇ is a linear transformation.

(c) Consider the function $c\colon VF \to VF$ defined by

$$c\left(\begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}\right) = \begin{pmatrix} \frac{\partial}{\partial y} F_3(x,y,z) - \frac{\partial}{\partial z} F_2(x,y,z) \\ \frac{\partial}{\partial z} F_1(x,y,z) - \frac{\partial}{\partial x} F_3(x,y,z) \\ \frac{\partial}{\partial x} F_2(x,y,z) - \frac{\partial}{\partial y} F_1(x,y,z) \end{pmatrix}$$

(This function is called the *curl* of the vector field) Prove that the function $c\colon VF \to VF$ is linear.

Remark: A good part of a course in vector calculus is concerned with studying the relationship between the image of the gradient and the kernel of curl. It turns out that the image of the gradient is always a subspace of the kernel of curl. In the previous homework, in the extra-credit problem, you showed how it is possible to form a "quotient vector space". Forming the quotient vector space from the kernel of curl and the image of gradient is called the "first cohomology group" of the domain of the scalar fields and vector fields. In the case of $\mathbb{R}^3$, it is just the vector space consisting only of the 0 vector, but in general it measures the number of "1-dimensional holes" in the space. It is an important object in electromagnetism.